052600 VU
Signal and Image Processing

Fourier Transform 2: Introduction to 2D Fourier Transform

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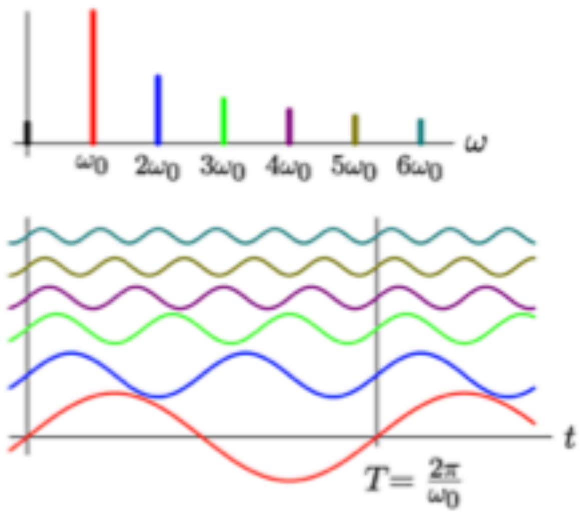
Recall - Fourier Series

- Fourier series is a way to represent a function as the sum of simple sine waves

- It decomposes any periodic signal into the sum of a set of simple sines and cosines (or complex exponentials)

- Recall Eulers formula:

$$e^{iy} = \cos(y) + i\sin(y)$$



Recall - Fourier Series

- Fourier Series is only applicable for periodic signals.
- Fourier Transform can be performed on aperiodic signals as well.
- It results in a function $F(\omega)$ that is continuous

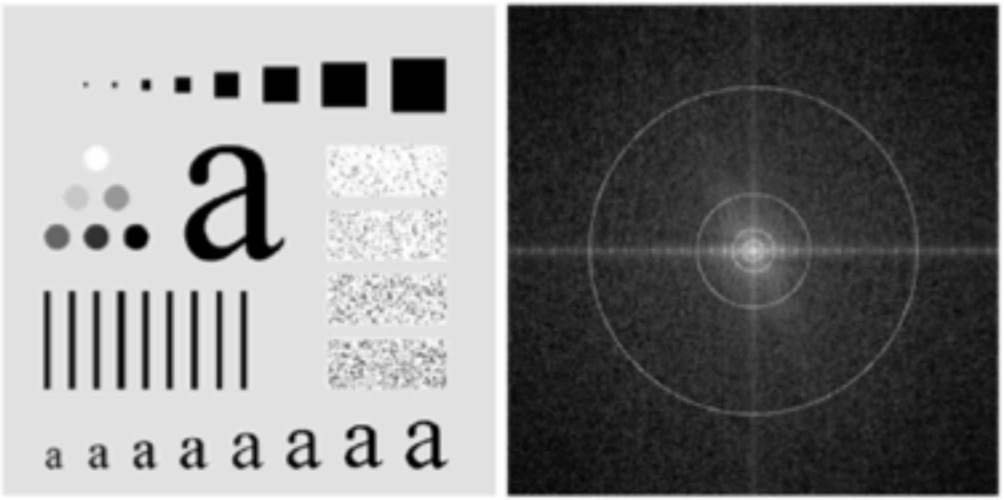
Overview

- Fourier Series (1D)
  - motivation
  - properties
  - examples
- Sampling & Impulse Train
- Fourier Transform (1D)
  - properties
  - convolution theorem
  - sampling in the Fourier space

All Fourier Transforms

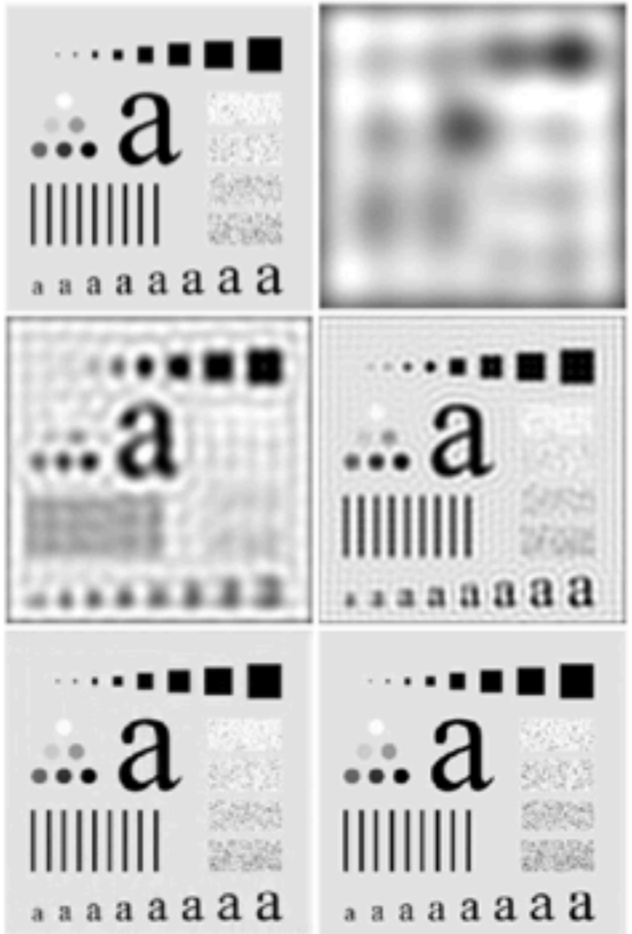
	Spatial Domain	Frequency Domain
FT	$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j2\pi\omega t} dt$ continuous	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\omega t} dt$ continuous
FS — Fourier Series	$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$ continuous + periodic	$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$ discrete
DFT — Discrete FT	$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$ discrete + periodic	$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$ discrete + periodic
DTFT — Discrete Time FT	$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega$ discrete	$F(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n} d\omega$ continuous + periodic

Some intuition



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



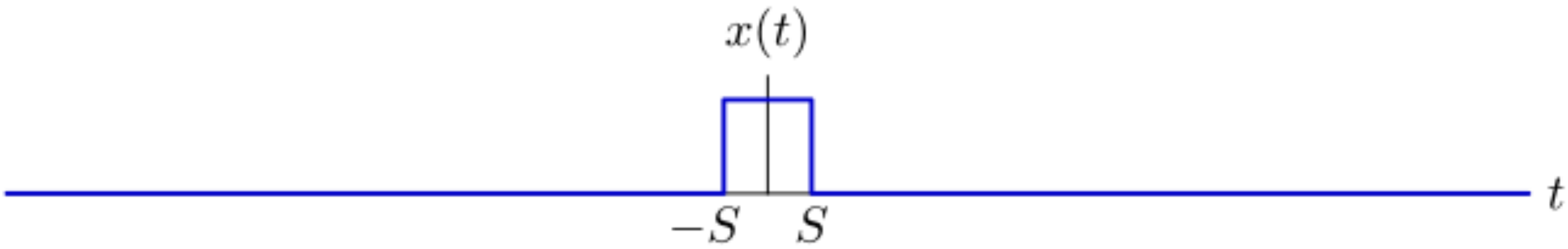
a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

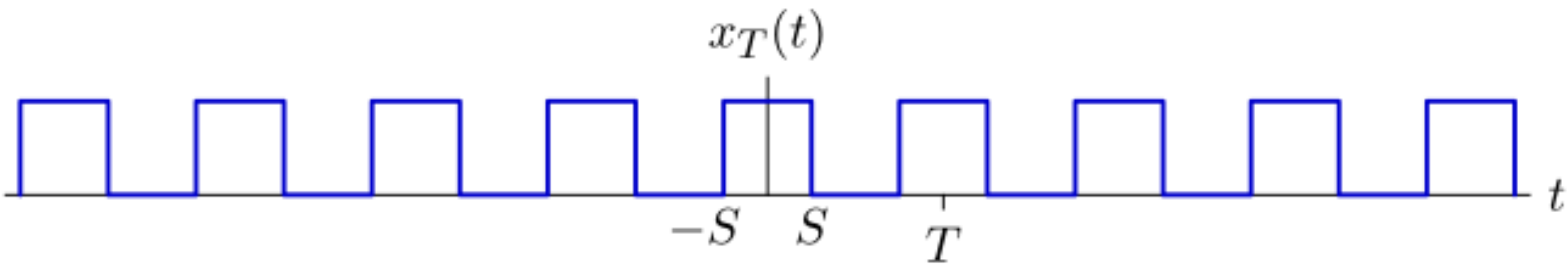
Fourier Transform

An aperiodic signal can be thought of as periodic with infinite period.

Let $x(t)$ represent an aperiodic signal.



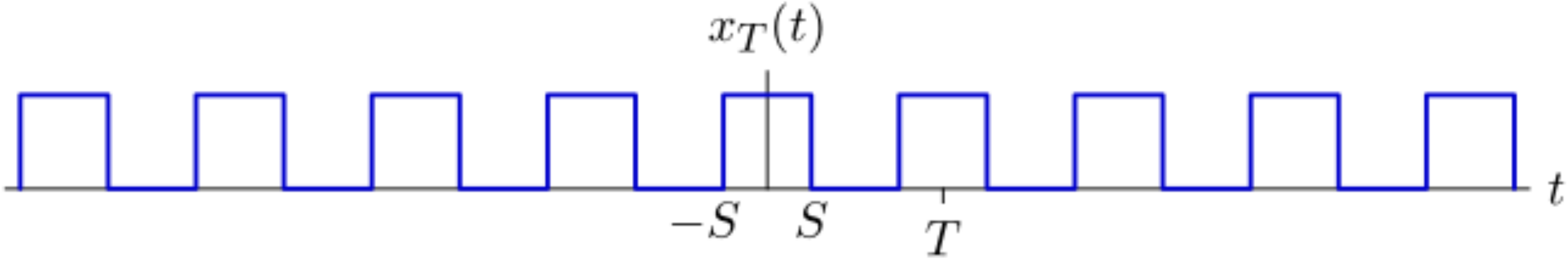
"Periodic extension": $x_T(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$



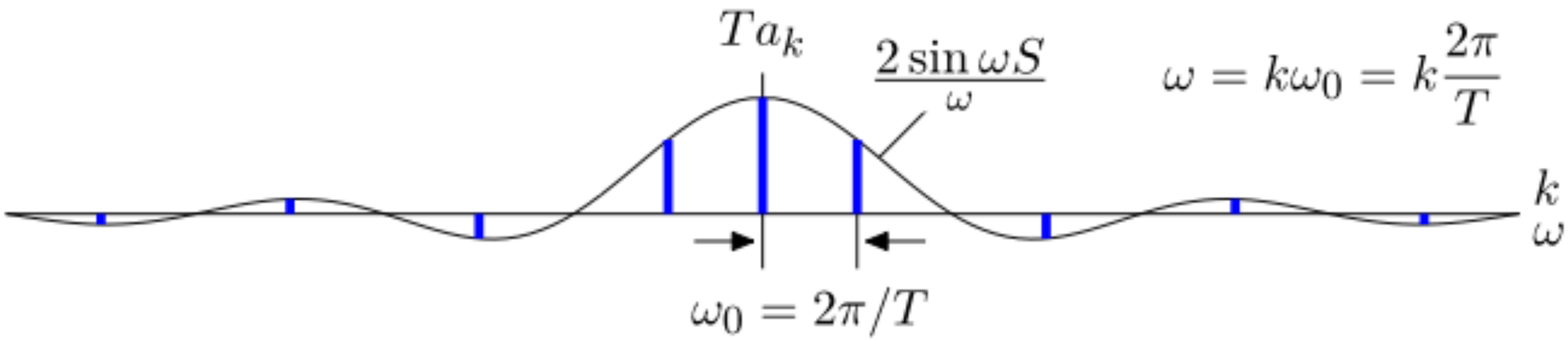
Then $x(t) = \lim_{T\to\infty} x_T(t)$.

Fourier Transform

Represent $x_T(t)$ by its Fourier series.

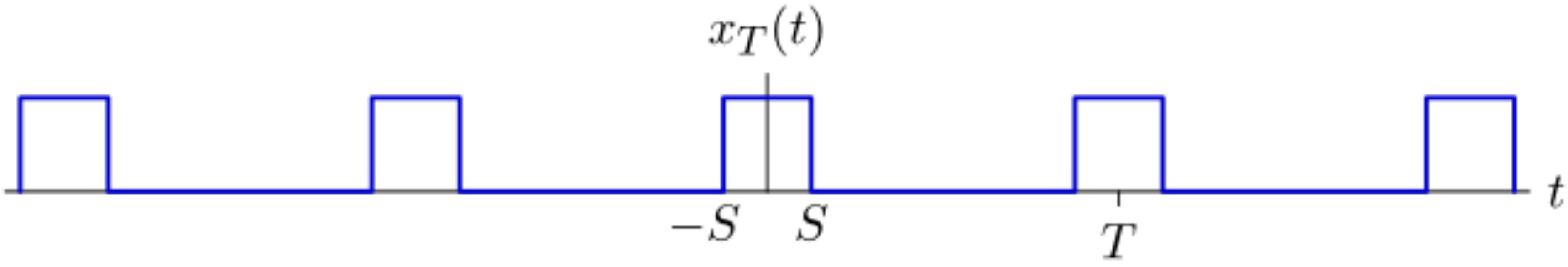


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

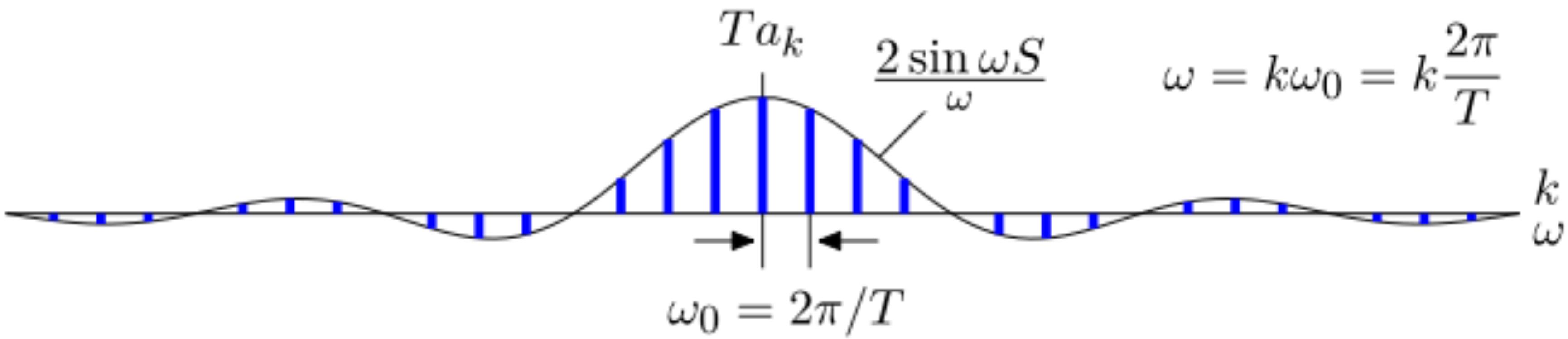


Fourier Transform

Doubling period doubles # of harmonics in given frequency interval.

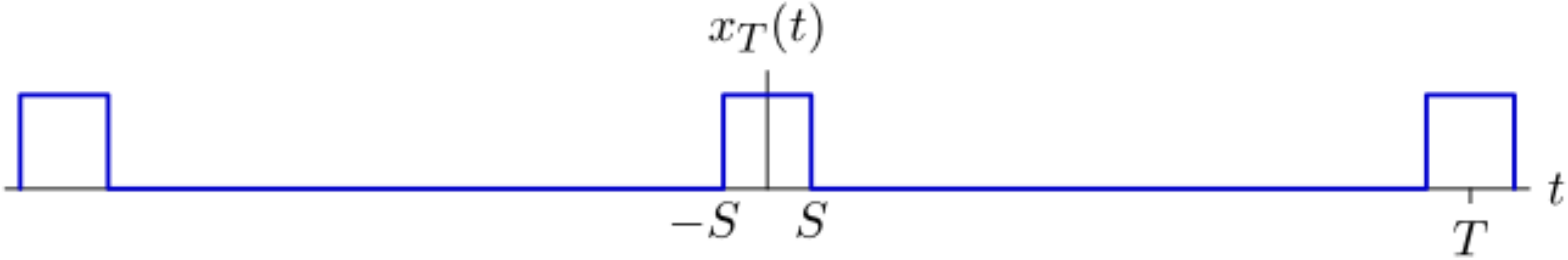


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

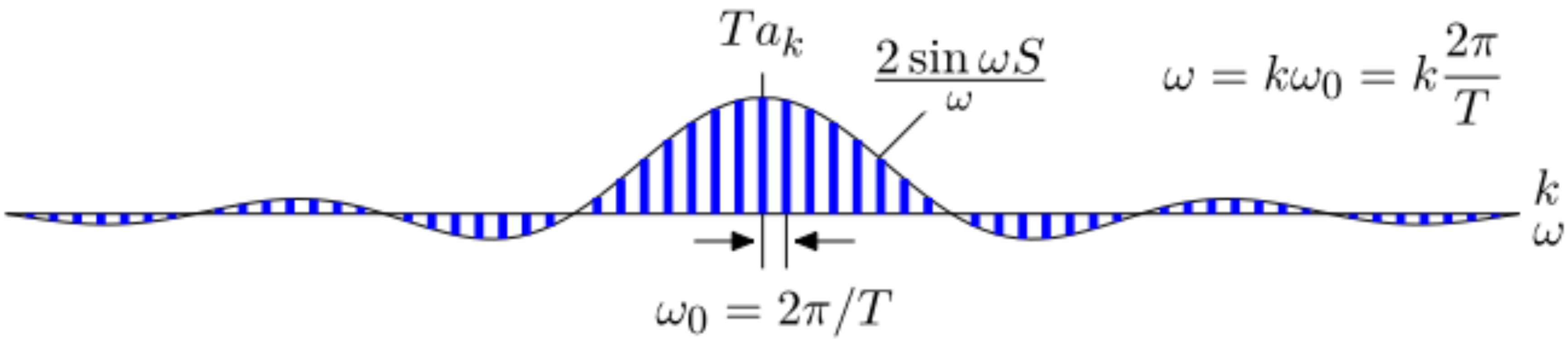


Fourier Transform

As $T \to \infty$, discrete harmonic amplitudes $\to$ a continuum $E(\omega)$.



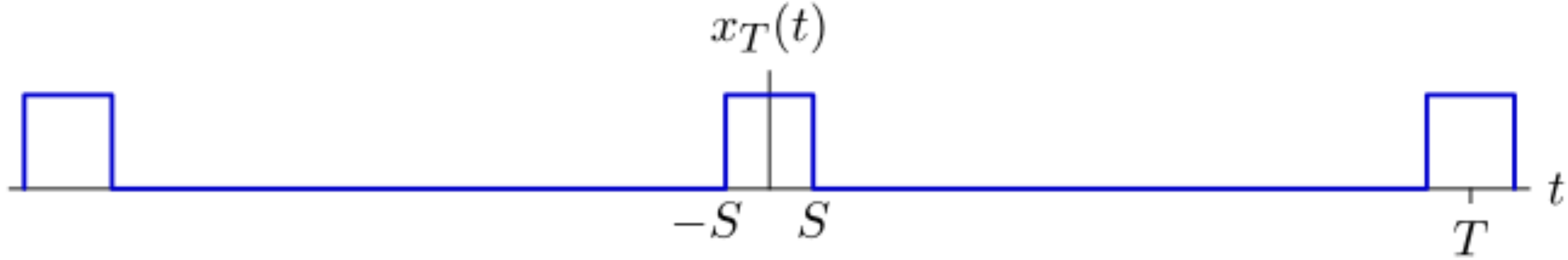
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

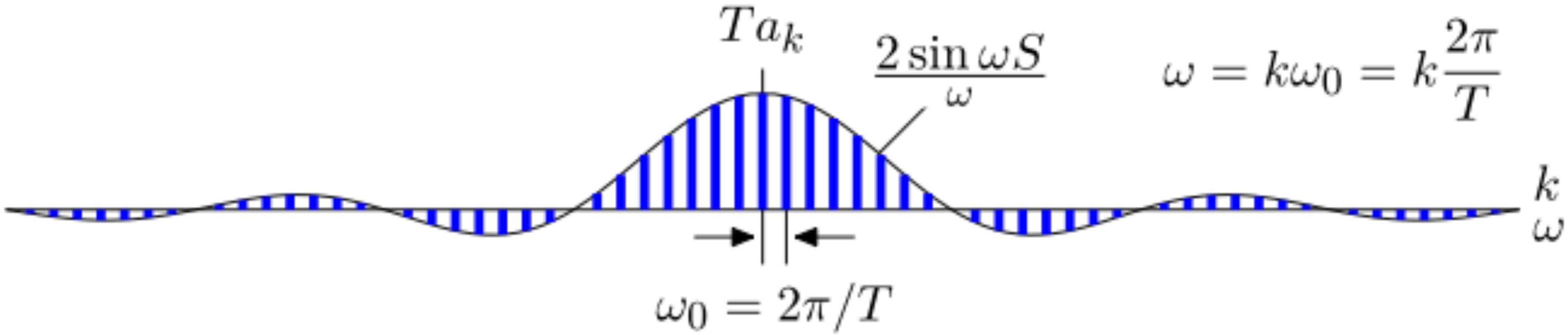


$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

Fourier Transform

As $T \to \infty$, synthesis sum $\to$ integral.





$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}E(\omega)}_{a_k} e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}E(\omega)e^{j\omega t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)e^{j\omega t}d\omega$$

Fourier Transform

Replacing $E(\omega)$ by $X(j\omega)$ yields the Fourier transform relations.

$$E(\omega) = X(j\omega)$$

Fourier transform

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(``analysis'' equation)}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{(``synthesis'' equation)}$$

Form is similar to that of Fourier series
$\rightarrow$ provides alternate view of signal.

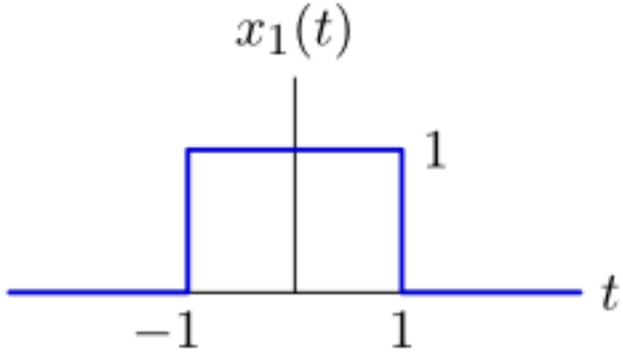
Properties of the Fourier Transform

Property	$x(t)$	$X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$
Time shift	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Time scale	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{j\omega}{a}\right)$
Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Multiply by t	$tx(t)$	$-\frac{1}{j}\frac{d}{d\omega}X(j\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) \times X_2(j\omega)$

Fourier Transform

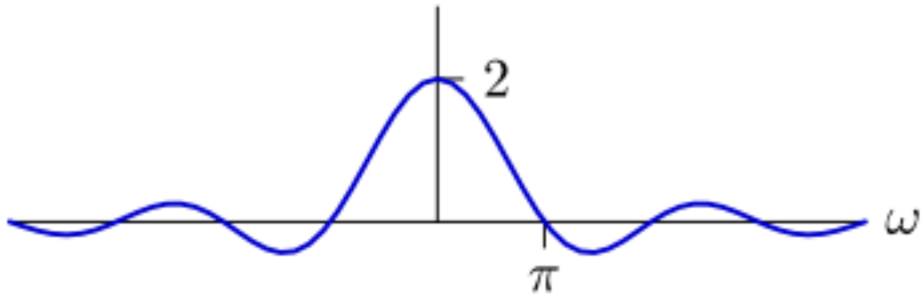
The Fourier transform is a function of real domain: frequency ω.

Time representation:



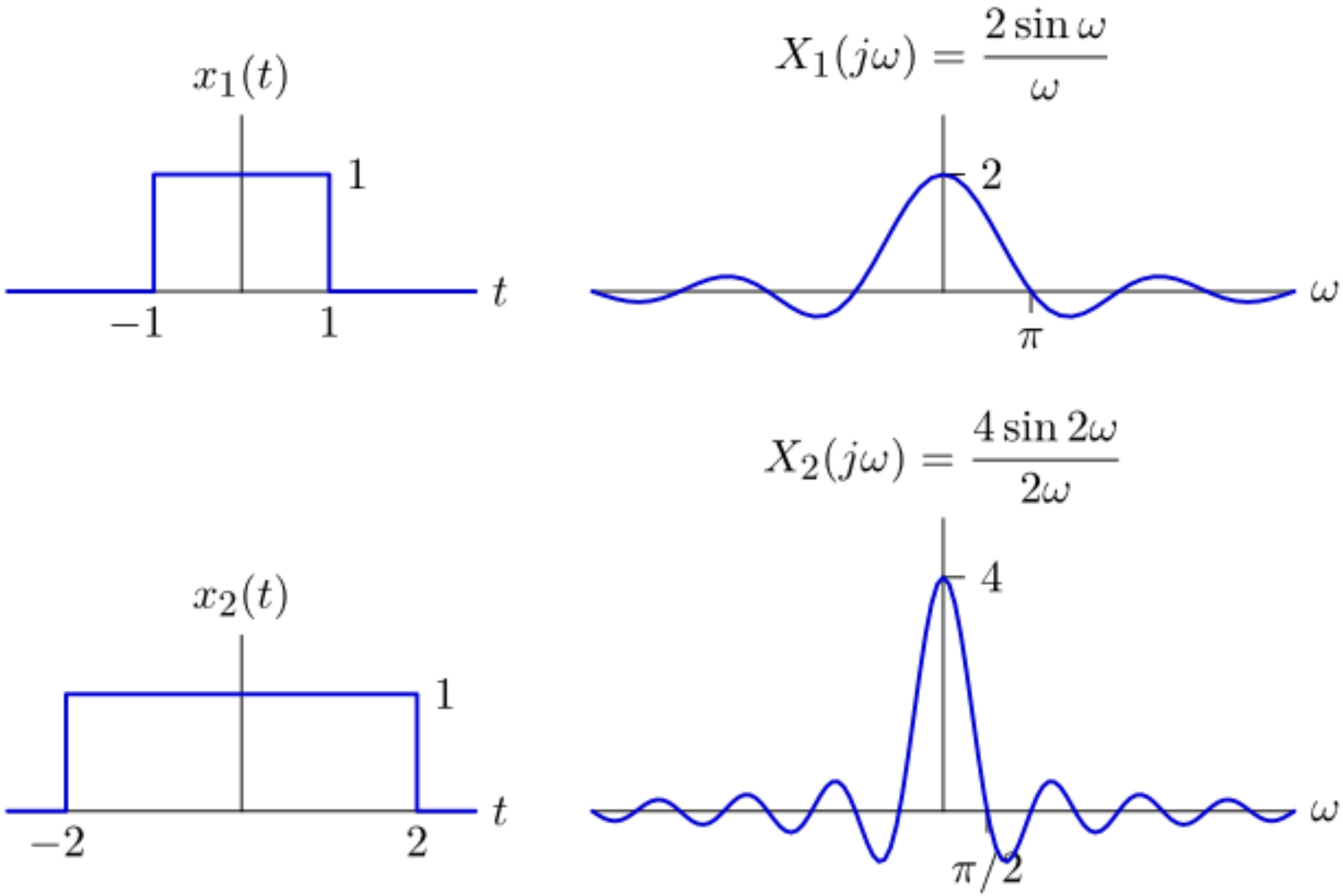
Frequency representation:

$$X_1(j\omega) = \frac{2\sin\omega}{\omega}$$



Fourier Transforms

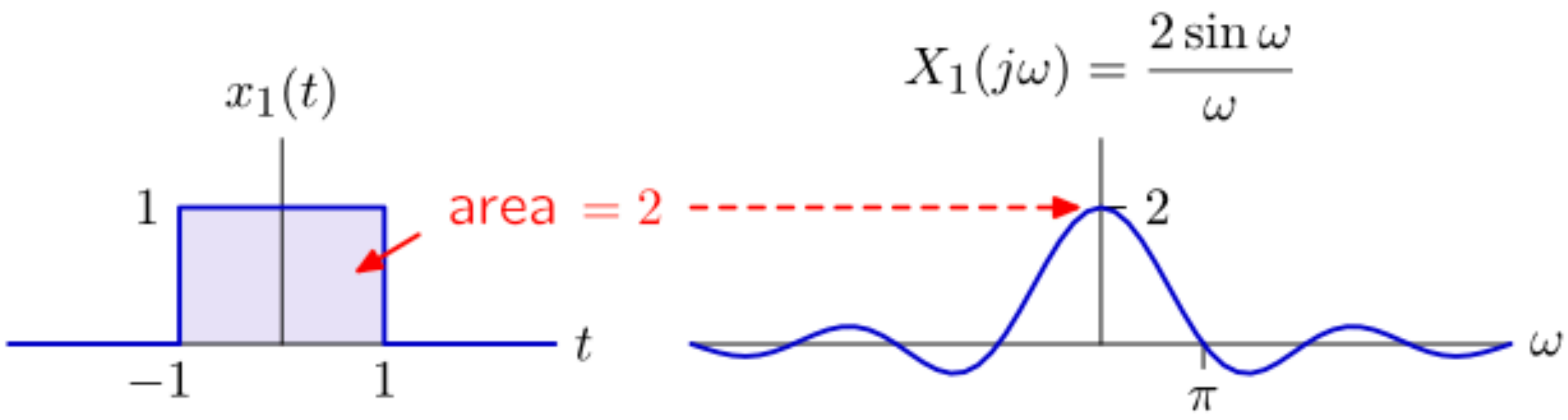
Stretching time compresses frequency.



Moments

The value of $X(j\omega)$ at $\omega = 0$ is the integral of $x(t)$ over time t.

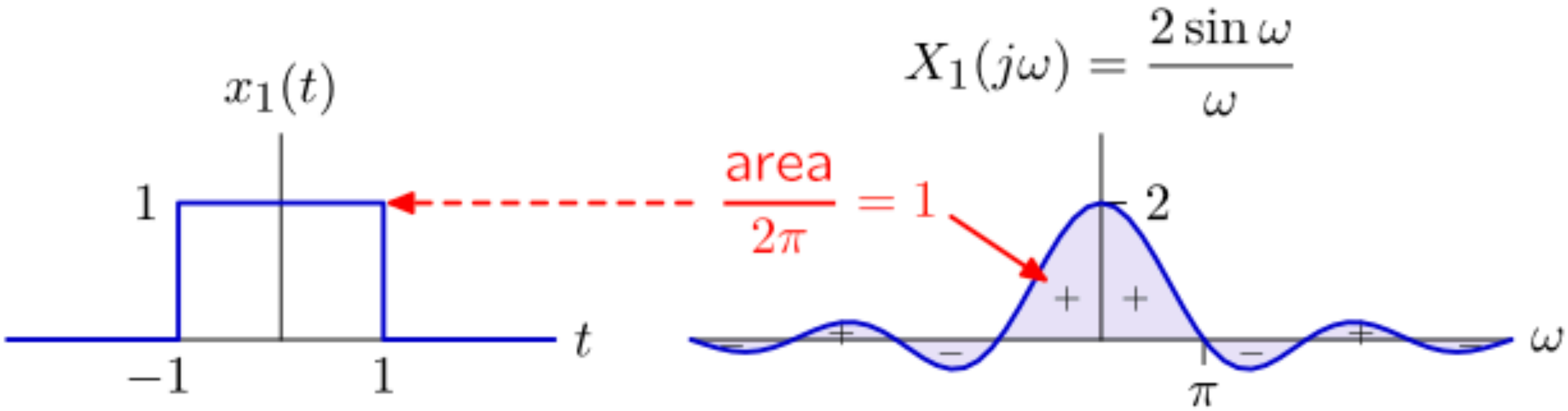
$$X(j\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j0t}dt = \int_{-\infty}^{\infty} x(t)\,dt$$



Moments

The value of $x(0)$ is the integral of $X(j\omega)$ divided by 2π.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$



Moments

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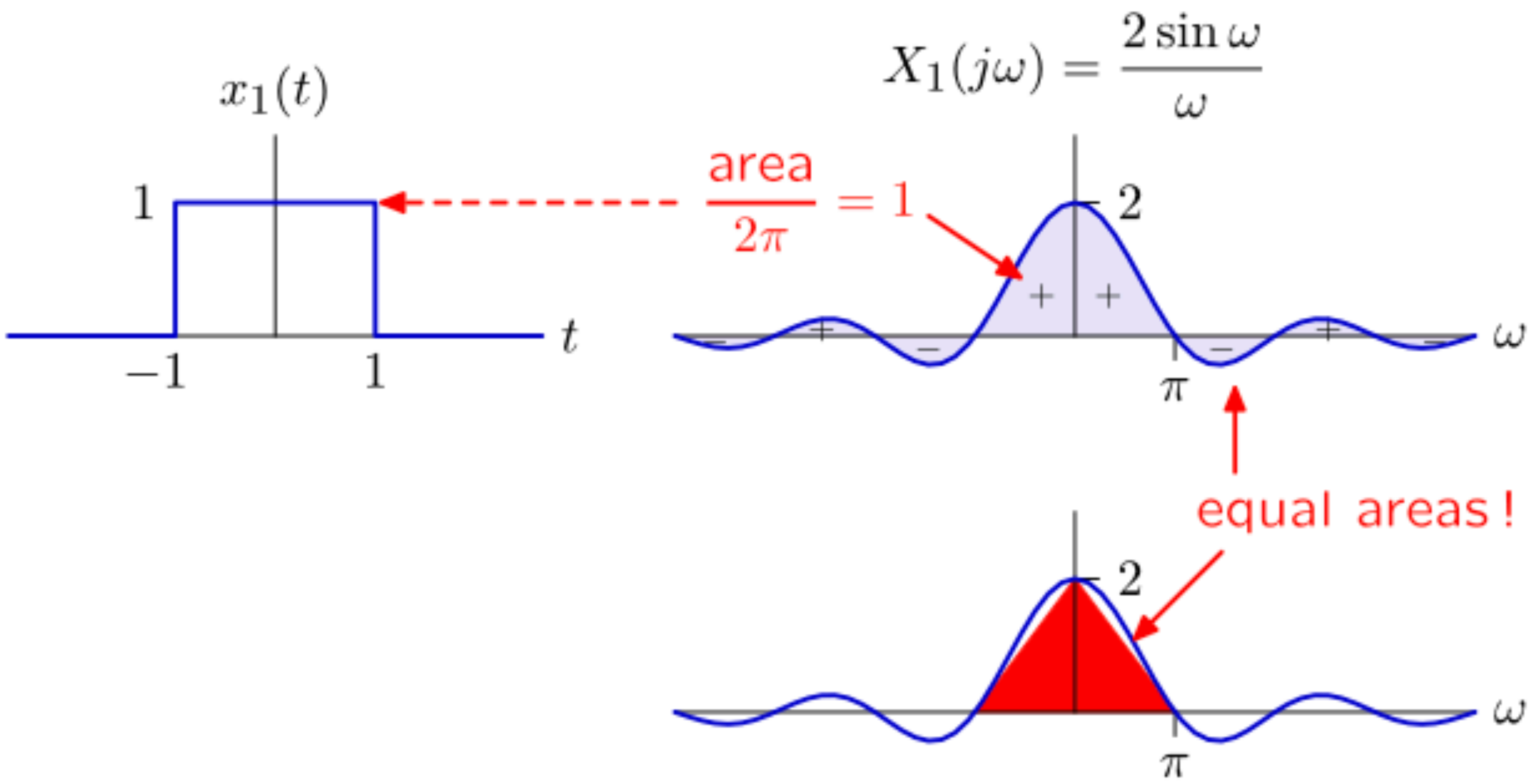
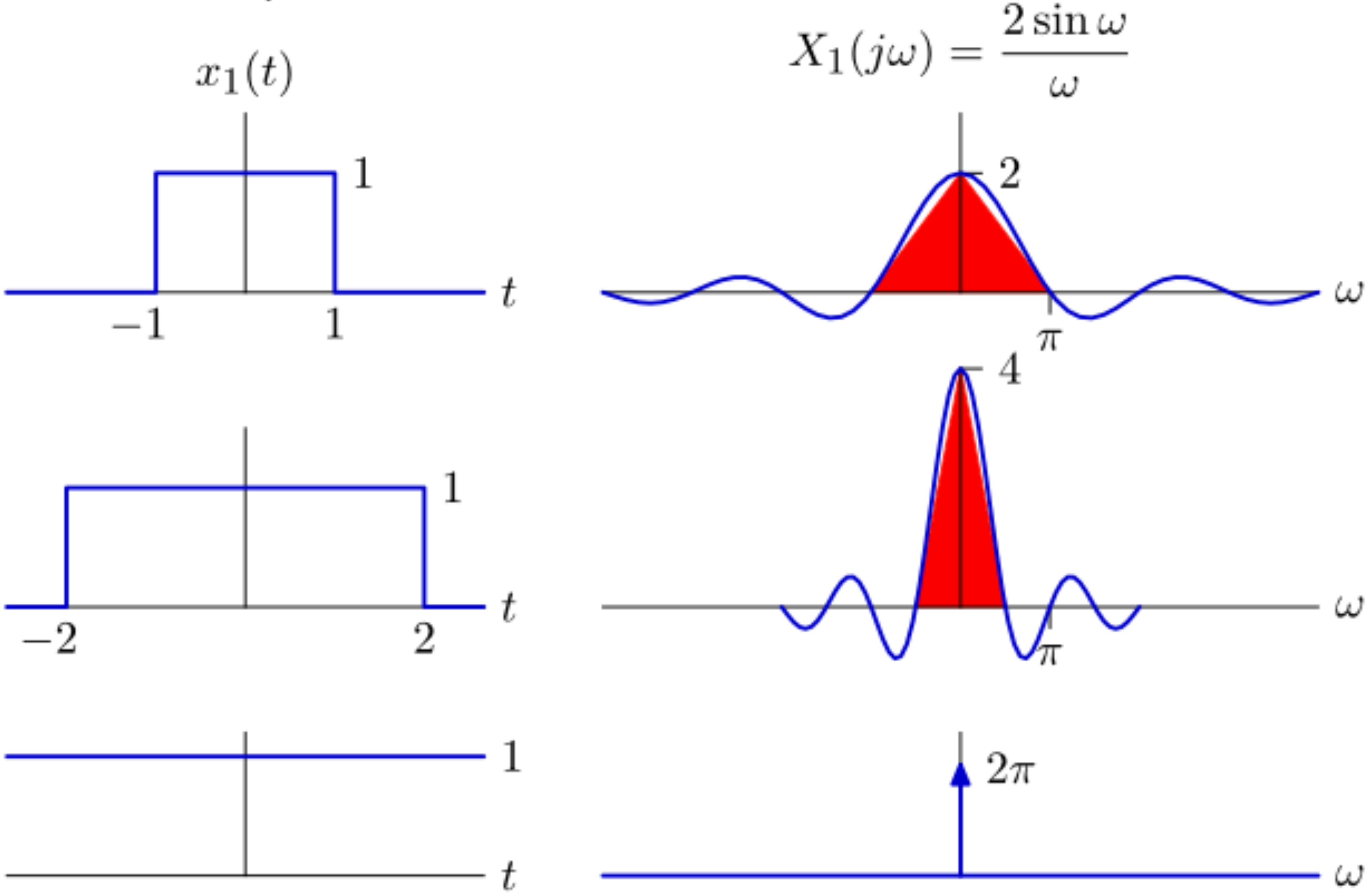


Stretching to the Limit

Stretching time compresses frequency and increases amplitude (preserving area).



New way to think about an impulse!

Fourier Transform

One of the most useful features of the Fourier transform (and Fourier series) is the simple "inverse" Fourier transform.

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Fourier transform)}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{("inverse" Fourier transform)}$$

There are 4 Fourier Transforms!

- Recall Fourier series:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}$$

$$f(t) \approx \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

- f(t) is periodic with period T!
- General Fourier Transform requires no periodicity:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j2\pi\omega t}dt$$

DFT — the most important one

- Discrete Fourier Transform (DFT) requires periodicity in both transform pairs

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$$

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$$

All Fourier Transforms

	Spatial Domain	Frequency Domain
FT	$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j2\pi\omega t} dt$ continuous	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\omega t} dt$ continuous
FS — Fourier Series	$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$ continuous + periodic	$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$ discrete
DFT — Discrete FT	$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$ discrete + periodic	$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$ discrete + periodic
DTFT — Discrete Time FT	$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega$ discrete	$F(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n} d\omega$ continuous + periodic

A little more intuition

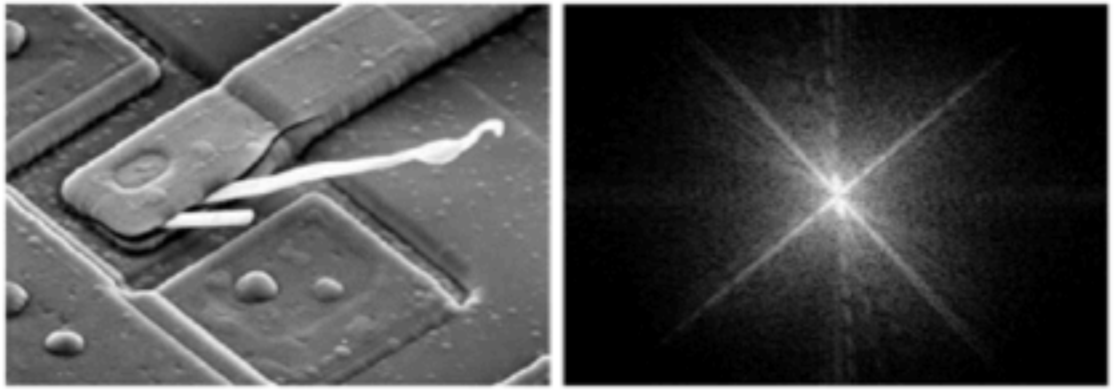


FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

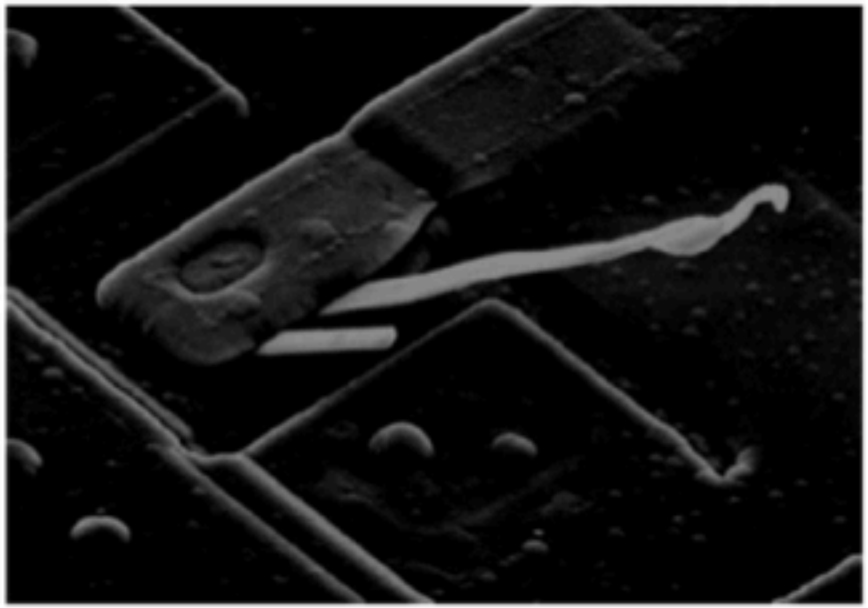
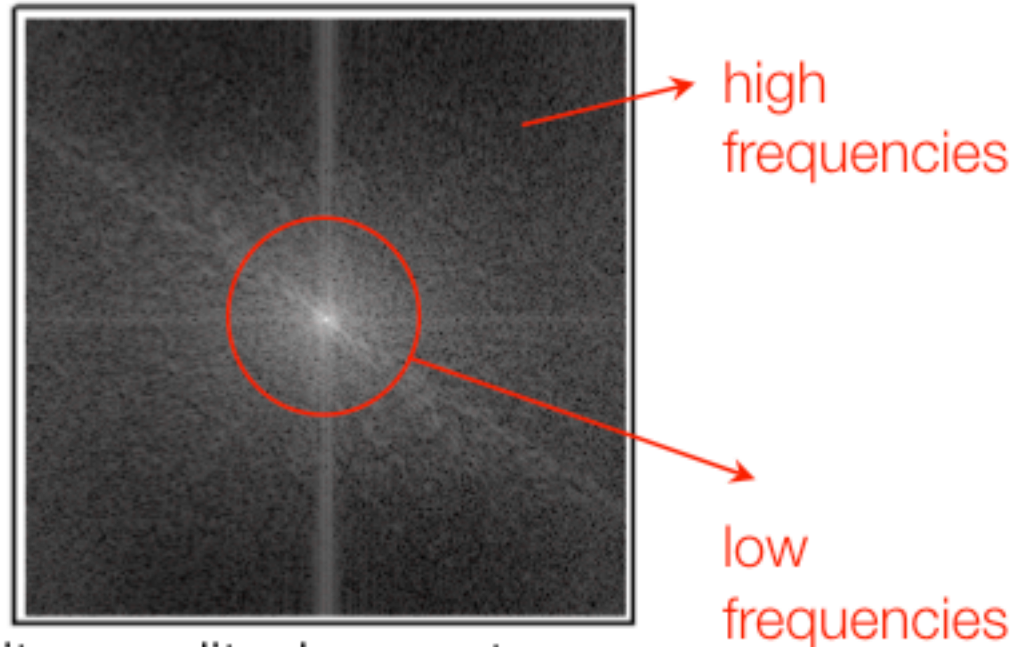


FIGURE 4.30 Result of filtering the image in Fig. 4.29(a) by setting to 0 the term $F(M/2, N/2)$ in the Fourier transform.

Visualisation of the spectrum

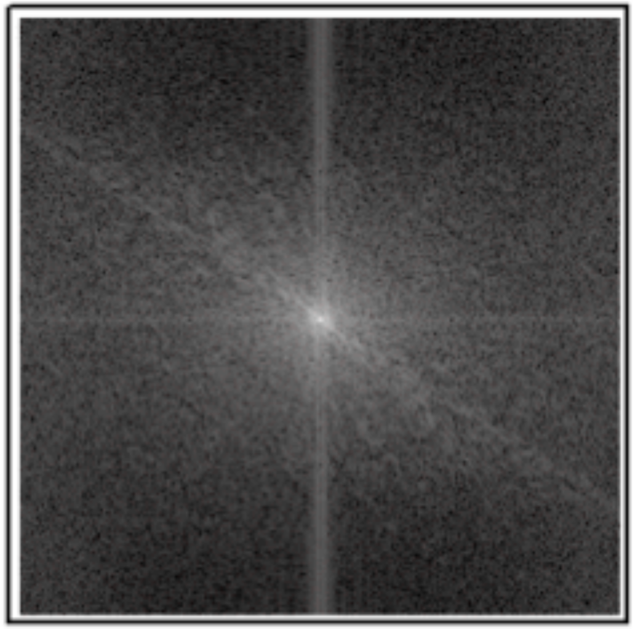
- Generally, we look at the amplitude of an image transform; hence we take a logarithmic scale to represent the values as gray-values.

original image



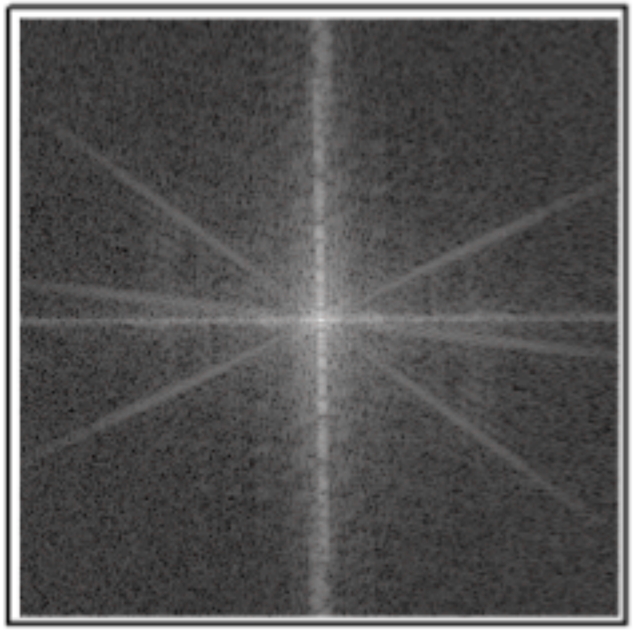
its amplitude spectrum

Interpretation



- The lines correspond to discontinuities, with perpendicular orientation.
- The horizontal and vertical lines come from the implicit periodic boundary conditions.
- For natural images, most of the information is concentrated in the low-frequency region.
- Low frequencies correspond to the slowly varying components, whereas high frequencies correspond to fast gray level changes (edges...)
- The diagonal line results from the discontinuity induced by the hat.

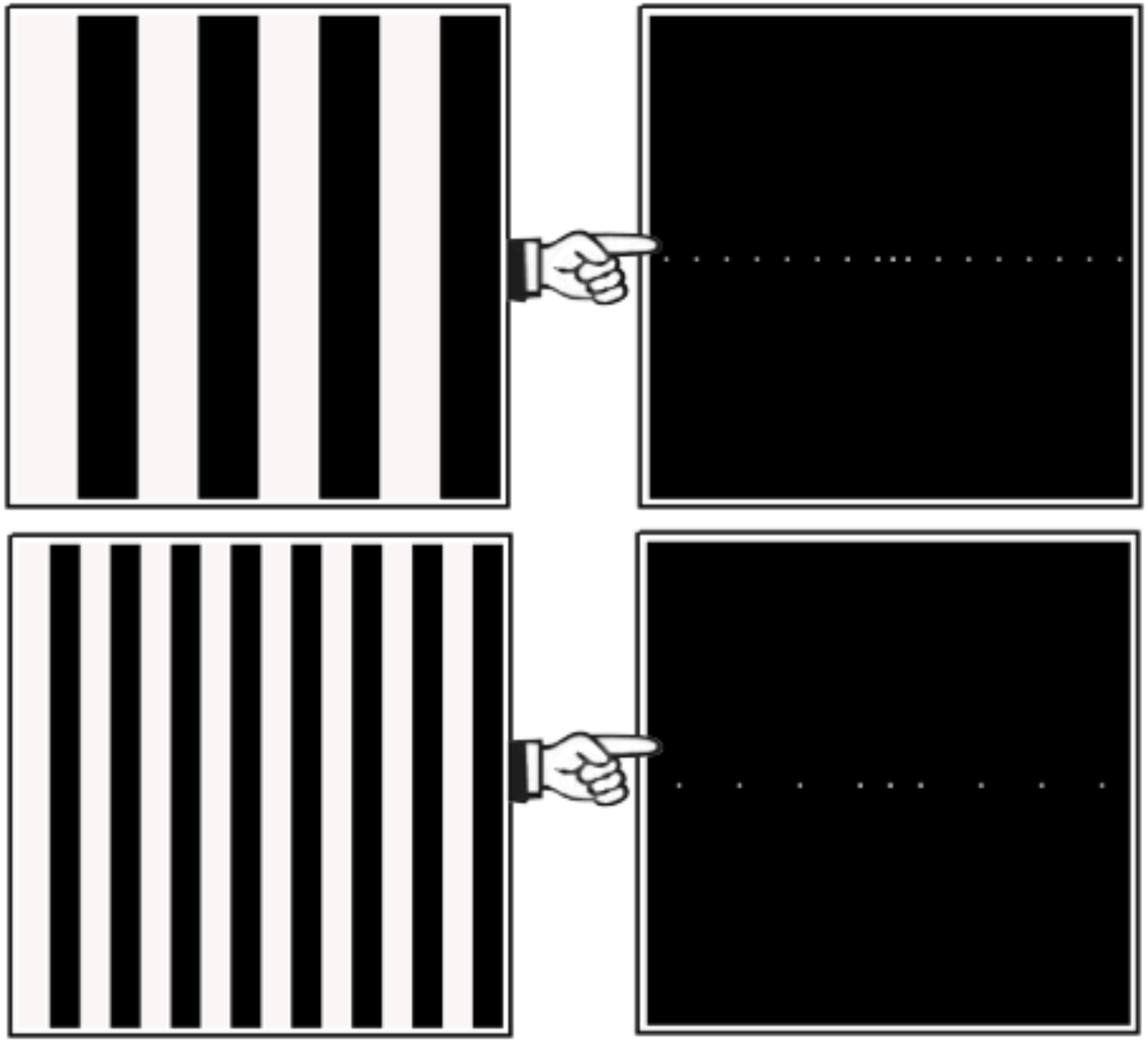
Interpretation



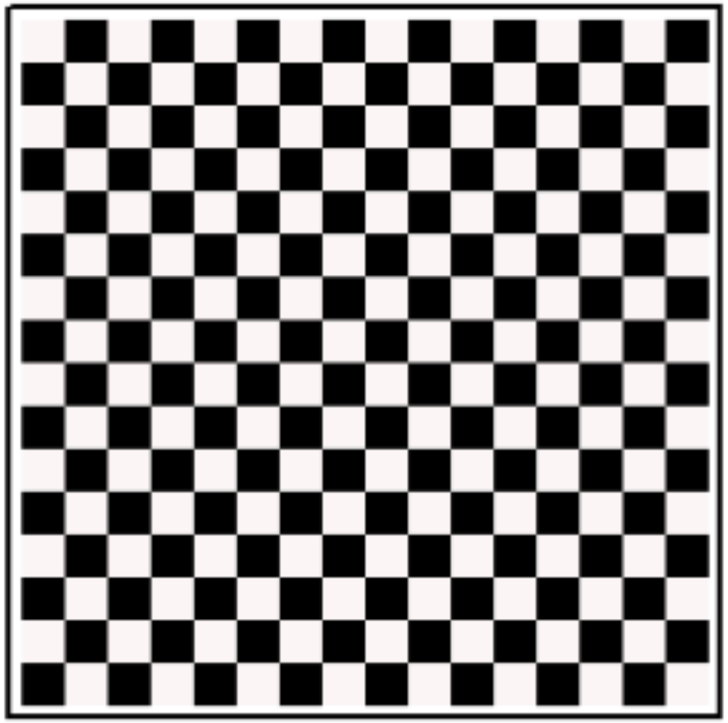
- The lines correspond to discontinuities, with perpendicular orientation.
- The horizontal and vertical lines come from the implicit periodic boundary conditions.
- For natural images, most of the information is concentrated in the low-frequency region.
- Low frequencies correspond to the slowly varying components, whereas high frequencies correspond to fast gray level changes (edges...)

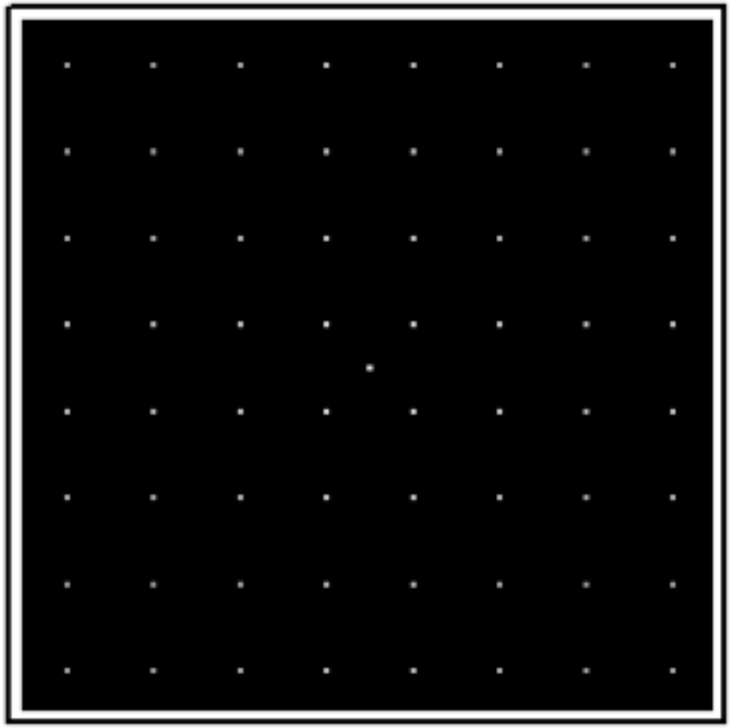
Some Examples

- Since the image content is periodic, the spectrum is discrete.
- If the period of the signal increases, the distance between the frequencies decreases, and vice versa.



Some Examples

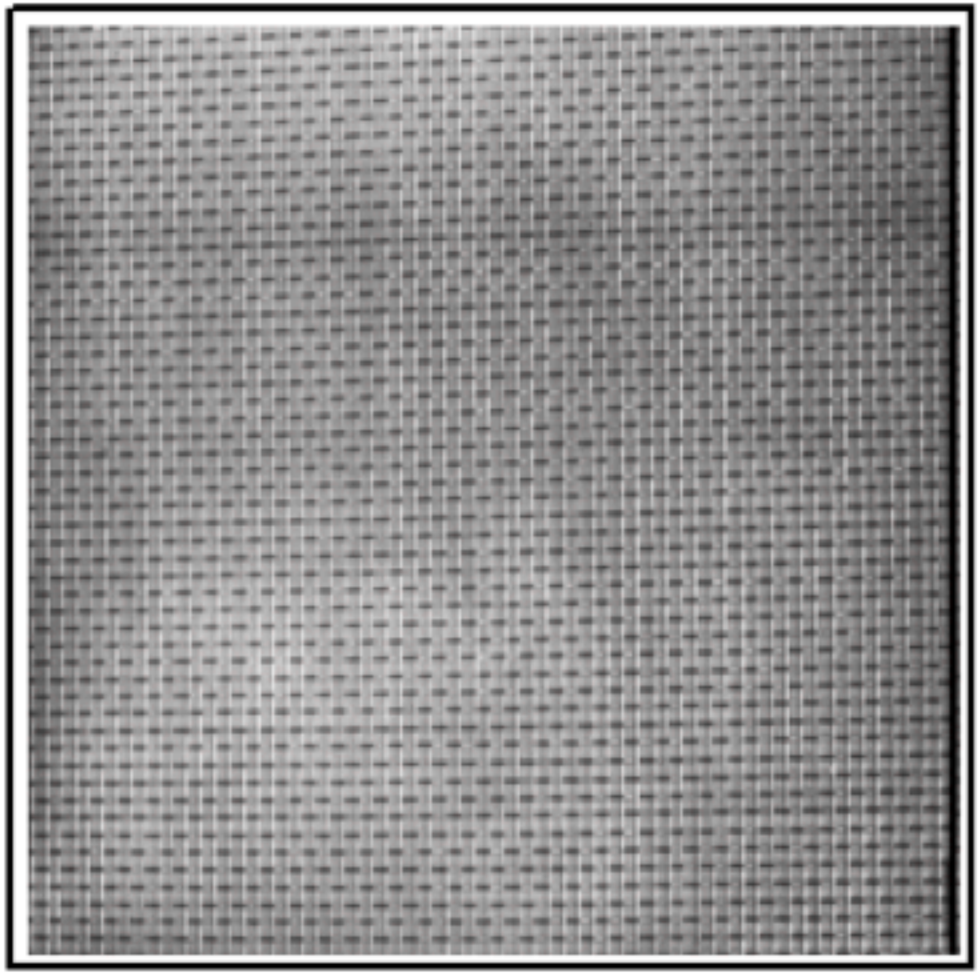


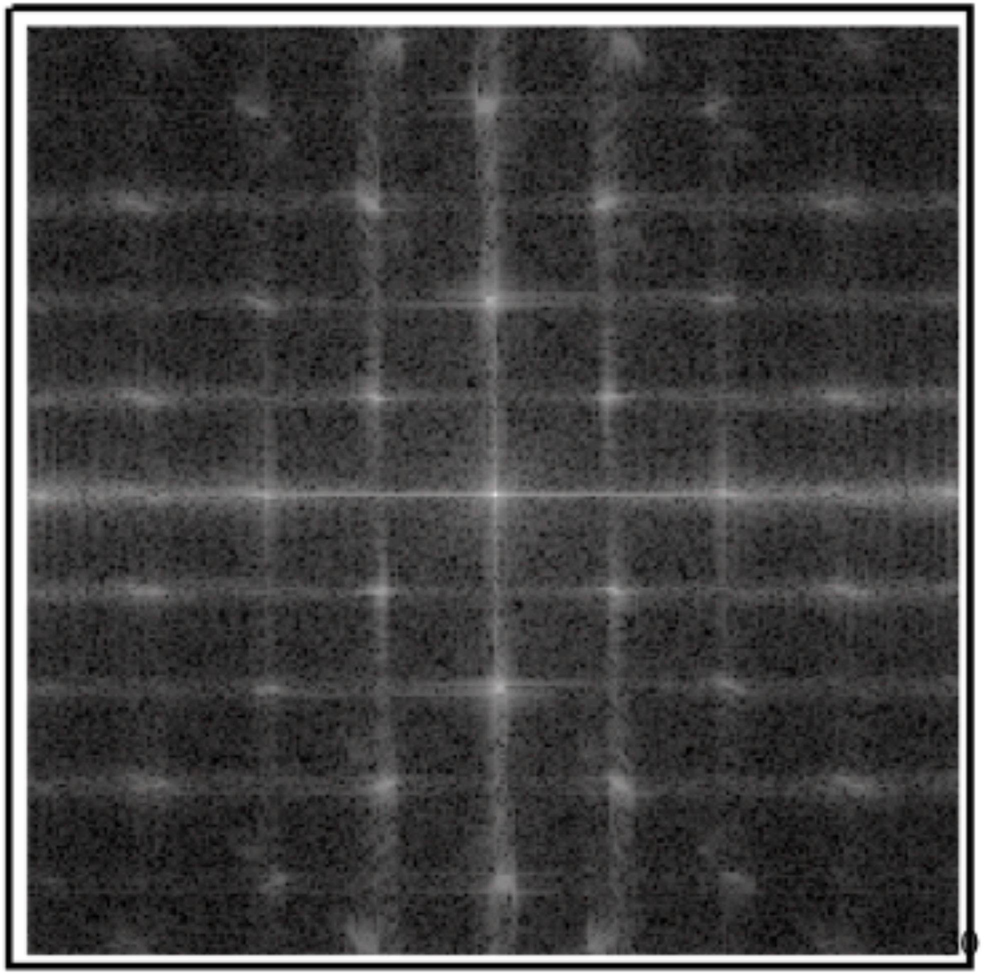


- The 2D periodicity of the image induces the 2D periodicity of the FT
- Since the image content is periodic, the spectrum is discrete.

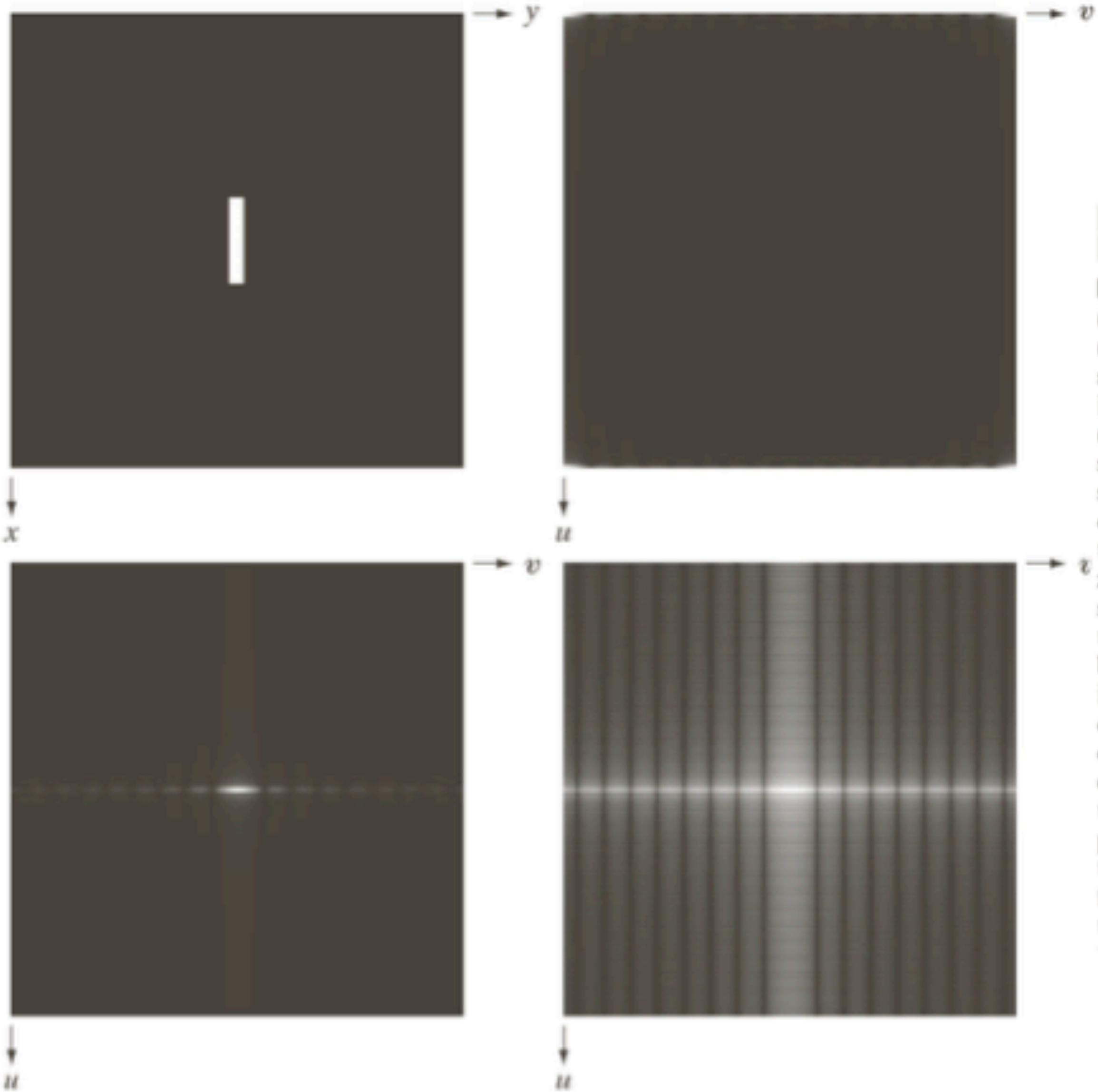
Texture = periodic pattern

- The Fourier spectrum is well suited for describing the directionality of textures





A little more intuition



a b
c d

FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

A little more intuition

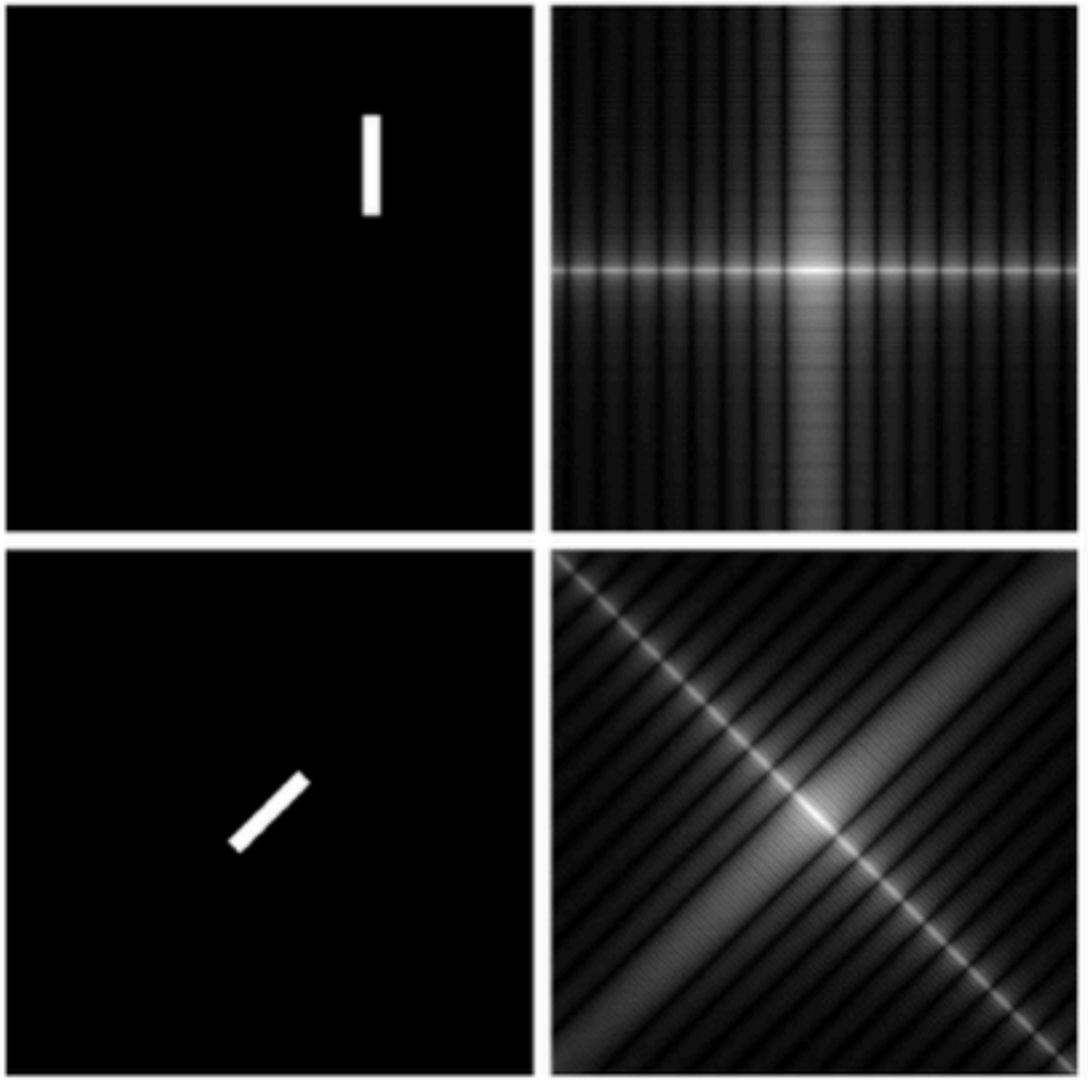
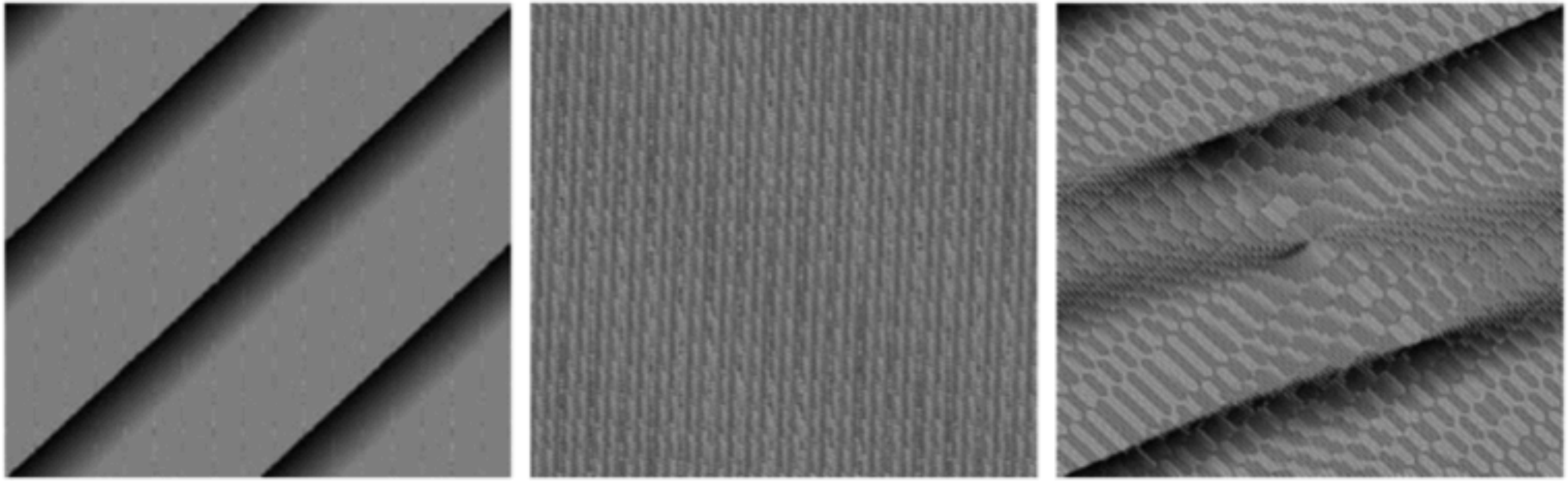


FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

A little more intuition



a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

Not very intuitive

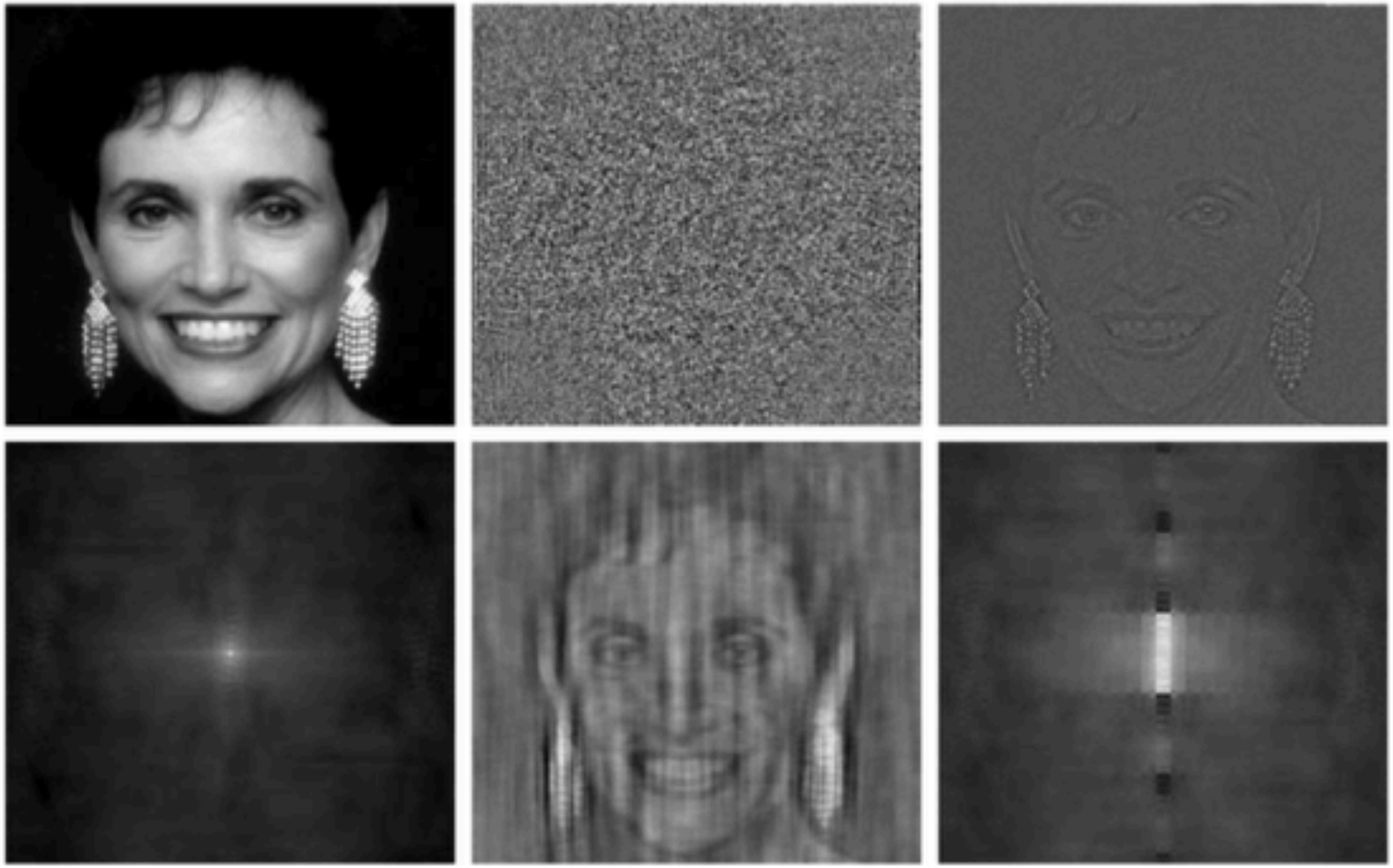


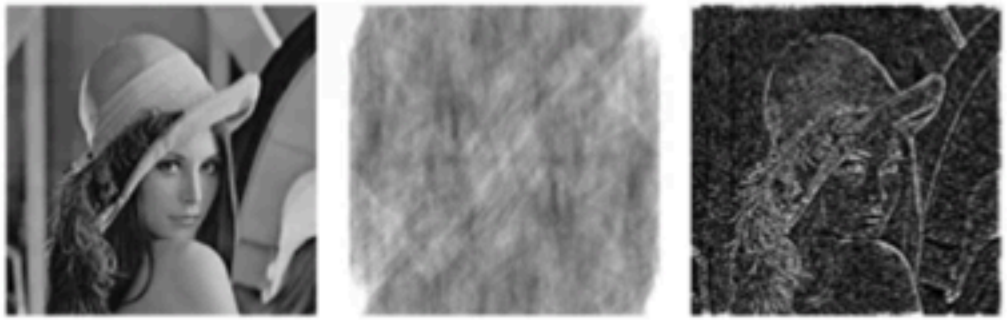
FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Learning Goals

- What are the differences between the Fourier Series and the Fourier Transform?
- Why is the Fourier Transform so important?
- Low frequencies correspond to …. in an image?
- High frequencies correspond to …. in an image?

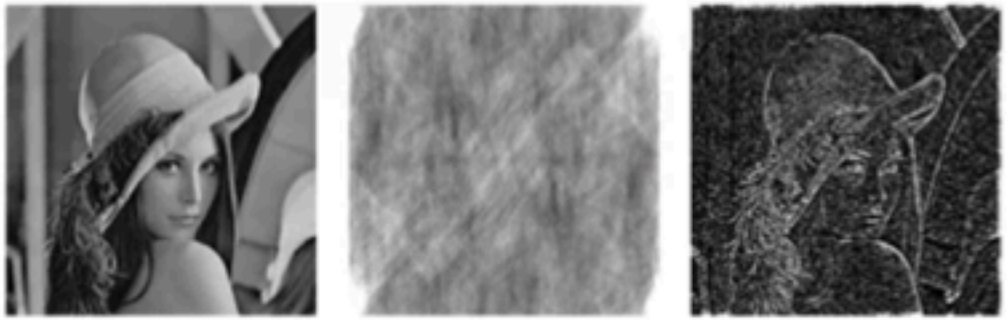
Recall

- What are the properties of $F(\omega)$ in the Fourier Transform?
- Which information contains the amplitude of a spectrum and which information contains the phase of a spectrum?



Recall

- Amplitude spectrum alone (middle image) and phase spectrum alone (rightmost image)
- with the phase spectrum alone, you can make sense of the nature of the image, even though the details are unclear.
- With the amplitude spectrum alone however, you cannot make any sense of the image, even though the dimensions(relative size of each component) are given.



Overview

- Fourier Series (1D)
  - motivation
  - properties
  - examples
- Sampling & Impulse Train
- Fourier Transform (1D)
  - properties
  - convolution theorem
  - sampling in the Fourier space

What happens to an impulse?

- it is basically a constant!

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T} e^0$$

$$c_n = \frac{1}{T}$$

What about a shifted impulse?

- the shifts remain as frequencies

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t - t_0) e^{-j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T} e^{-j\frac{2\pi n}{T}t_0}$$

What happens to an impulse train?

- Impulse train is periodic — apply Fourier series, will not do the math here, see book:

$$s_{\Delta T}(t) = \sum_{-\infty}^{\infty} \delta(t - n\Delta T)$$

$$S_{\Delta T}(\omega) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{n}{\Delta T})$$

- distance between impulses grows inversely

What happens to a box?

- it is the well-known sinc function

$$\text{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

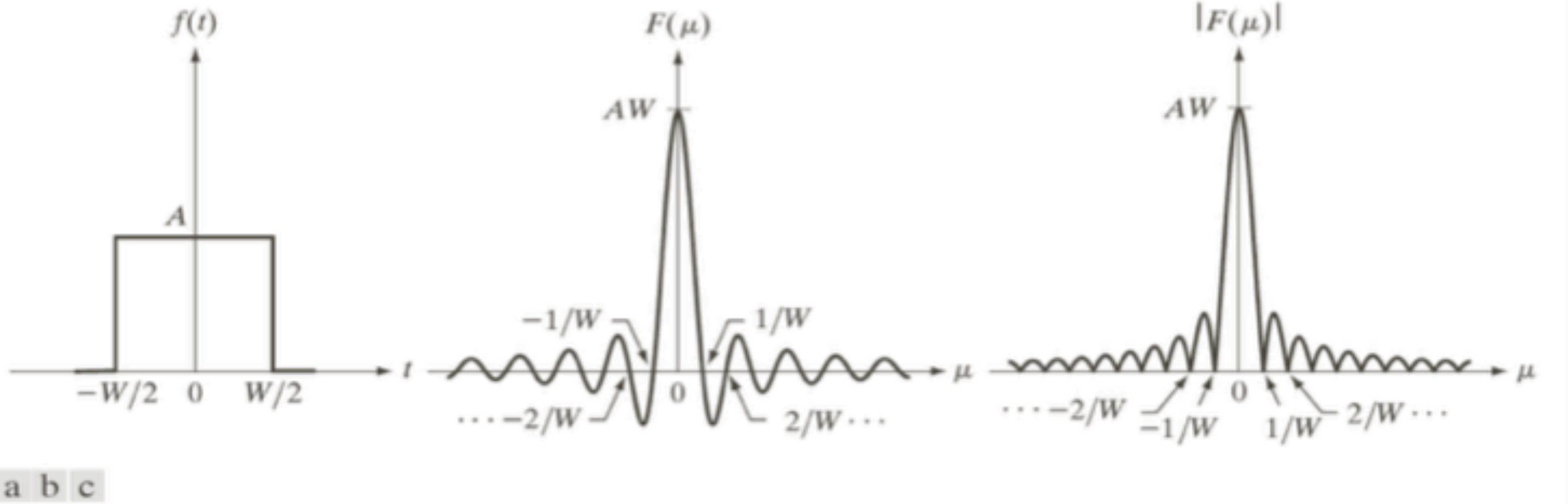


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Overview

- Fourier Series (1D)
  - motivation
  - properties
  - examples
- Sampling & Impulse Train
- Fourier Transform (1D)
  - properties
  - convolution theorem
  - sampling in the Fourier space

LTI systems

$$f_s(t) = \sum_{n=-\infty}^{\infty} f[n]\delta(t - n\Delta T)$$

$$f_r(t) = \sum_{n=-\infty}^{\infty} f[n]\phi(t - n\Delta T)$$

LTI systems

- sine+cosine are eigenfunctions of an LTI system (a convolution)
- The eigenvalue popping out is the Fourier Transform of the LTI

$$e^{j\omega n\Delta T}$$

$$e^{j\omega n\Delta T}\Phi(\omega)$$

$$\Phi(\omega) = \sum_{m=-\infty}^{\infty} e^{-mj\omega\Delta T}\phi(m\Delta T)$$

What is the Fourier Transform of a convolution?

- doing this in the continuous domain:

$$f_r(t) = f * \phi(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$\begin{aligned} F_r(\omega) &= \int \left[\int f(\tau)h(t-\tau)d\tau \right] e^{-j2\pi\omega t} dt \\ &= \int f(\tau) \left[\int h(t-\tau)e^{-j2\pi\omega t} dt \right] d\tau \\ &= \int f(\tau) \left[H(\omega)e^{-j2\pi\omega\tau} \right] d\tau \\ &= H(\omega) \int f(\tau)e^{-j2\pi\omega\tau} d\tau \\ &= H(\omega)F(\omega) \end{aligned}$$

What is the Fourier Transform of a convolution?

- convolution == multiplication:

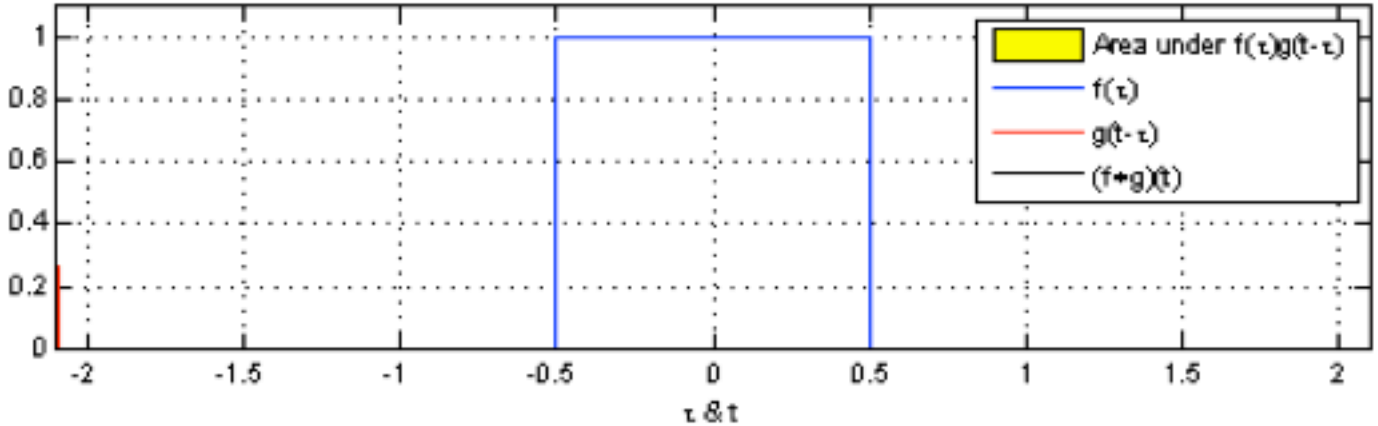
$$f * \phi(t) \Longleftrightarrow F(\omega)H(\omega)$$

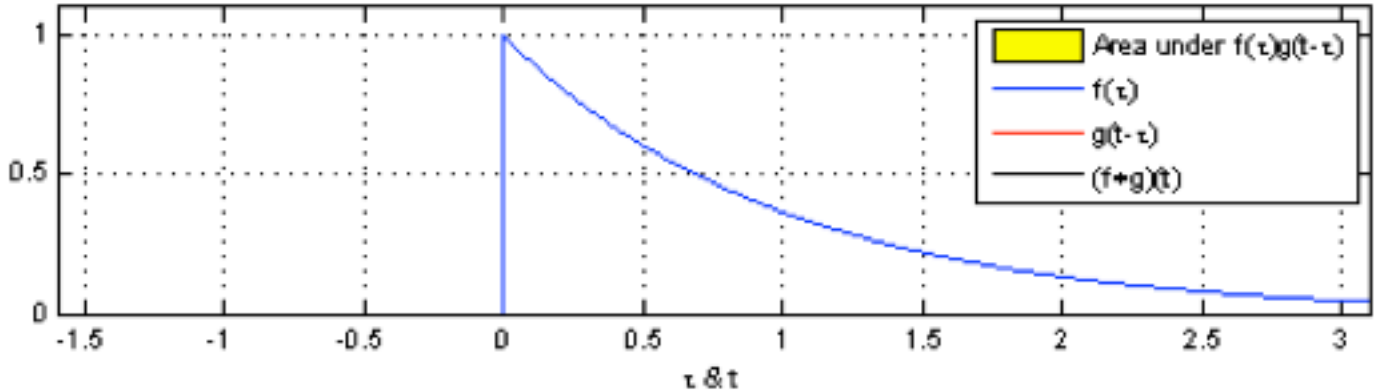
- multiplication == convolution:

$$f(t)\phi(t) \Longleftrightarrow F(\omega) * H(\omega)$$

Convolution — the movie

- from wikipedia:





Learning Goals

- Why is Convolution such an important concept?
- How does the Convolution work?
- What is an LTI System?
- How can I characterise an LTI System?

Overview

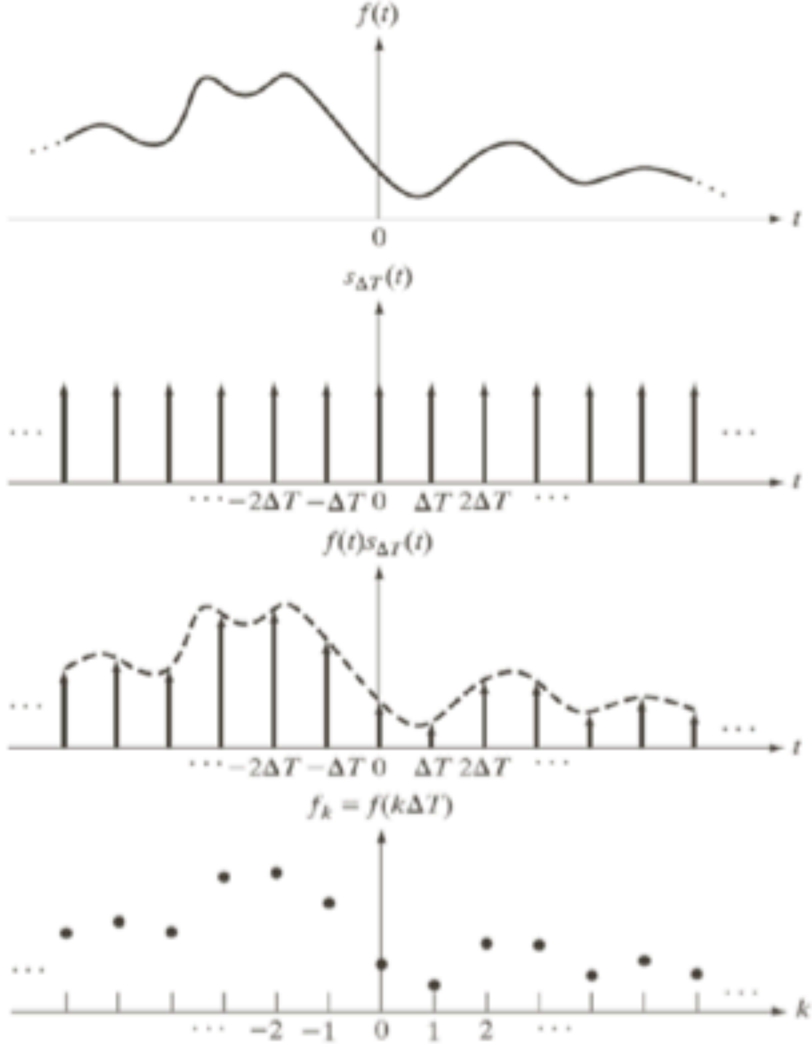
- Sampling & Impulse Train
- Fourier Transform (1D)
  - properties
  - convolution theorem
  - sampling in the Fourier space
- Fourier Transform (2D)

What was sampling again?

$$f(t) \times s_{\Delta T}(t) =$$

$$\sum_{-\infty}^{\infty} f(n\Delta T)\delta(t - n\Delta T)$$

$$\sum_{-\infty}^{\infty} f[n]\delta(t - n\Delta T)$$



Sampling in the Fourier Domain

$$f(t)s_{\Delta T}(t) \Longleftrightarrow F(\omega) * S_{\Delta T}(\omega)$$

$$F(\omega) * S_{\Delta T}(\omega) = F(\omega) * \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{n}{\Delta T})$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\omega - \frac{n}{\Delta T})$$

Sampling

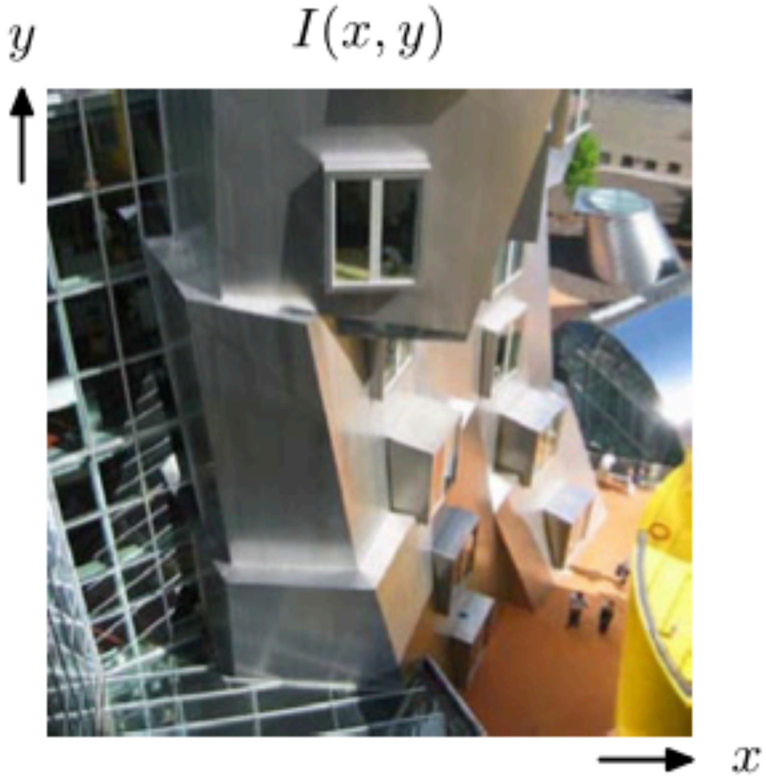
Sampling allows the use of modern digital electronics to process, record, transmit, store, and retrieve CT signals.

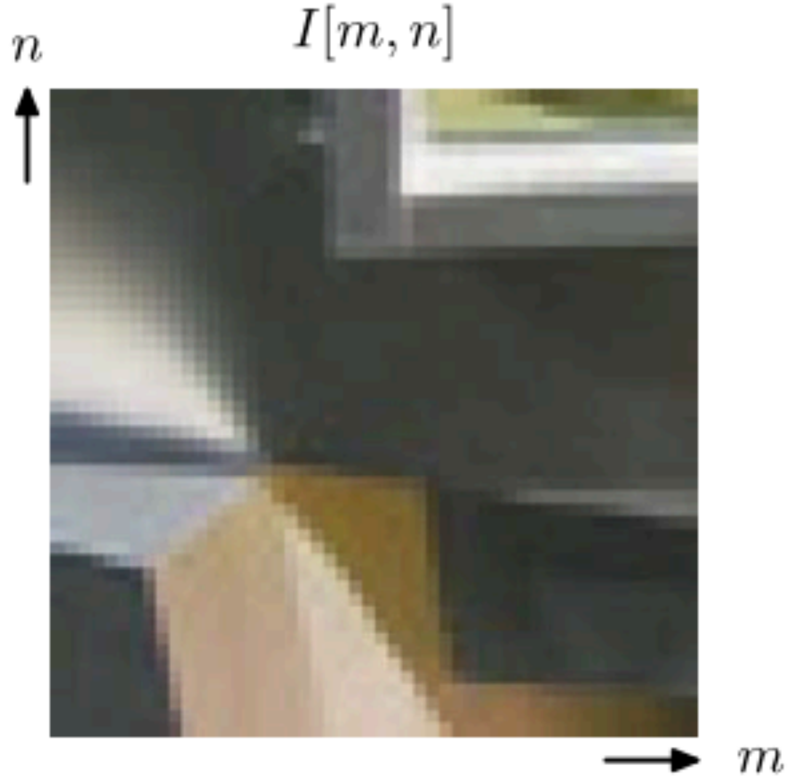
- audio: MP3, CD, cell phone
- pictures: digital camera, printer
- video: DVD
- everything on the web

Sampling

Sampling is pervasive.

Example: digital cameras record sampled images.



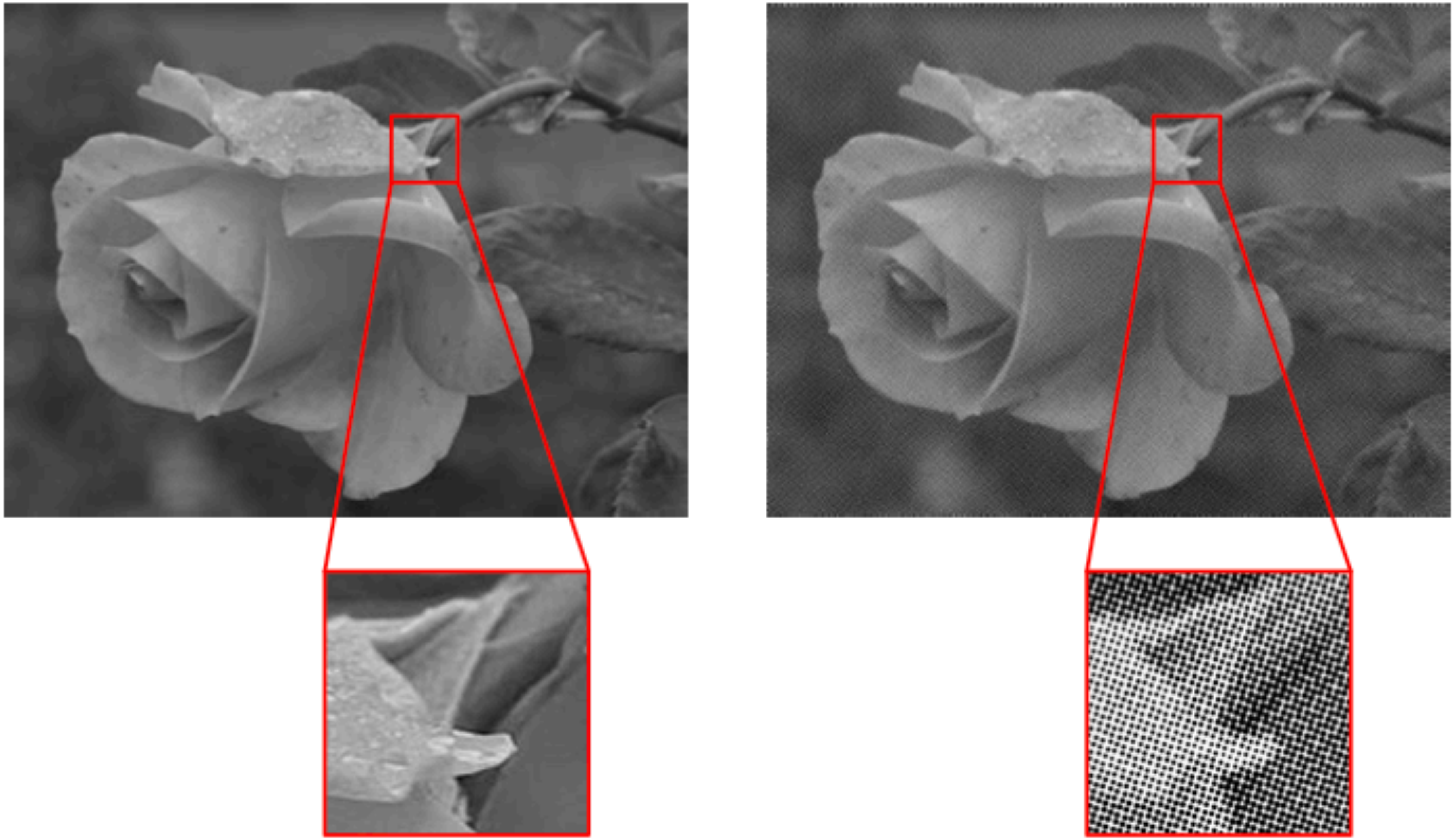


Sampling

Photographs in newsprint are "half-tone" images. Each point is black or white and the average conveys brightness.

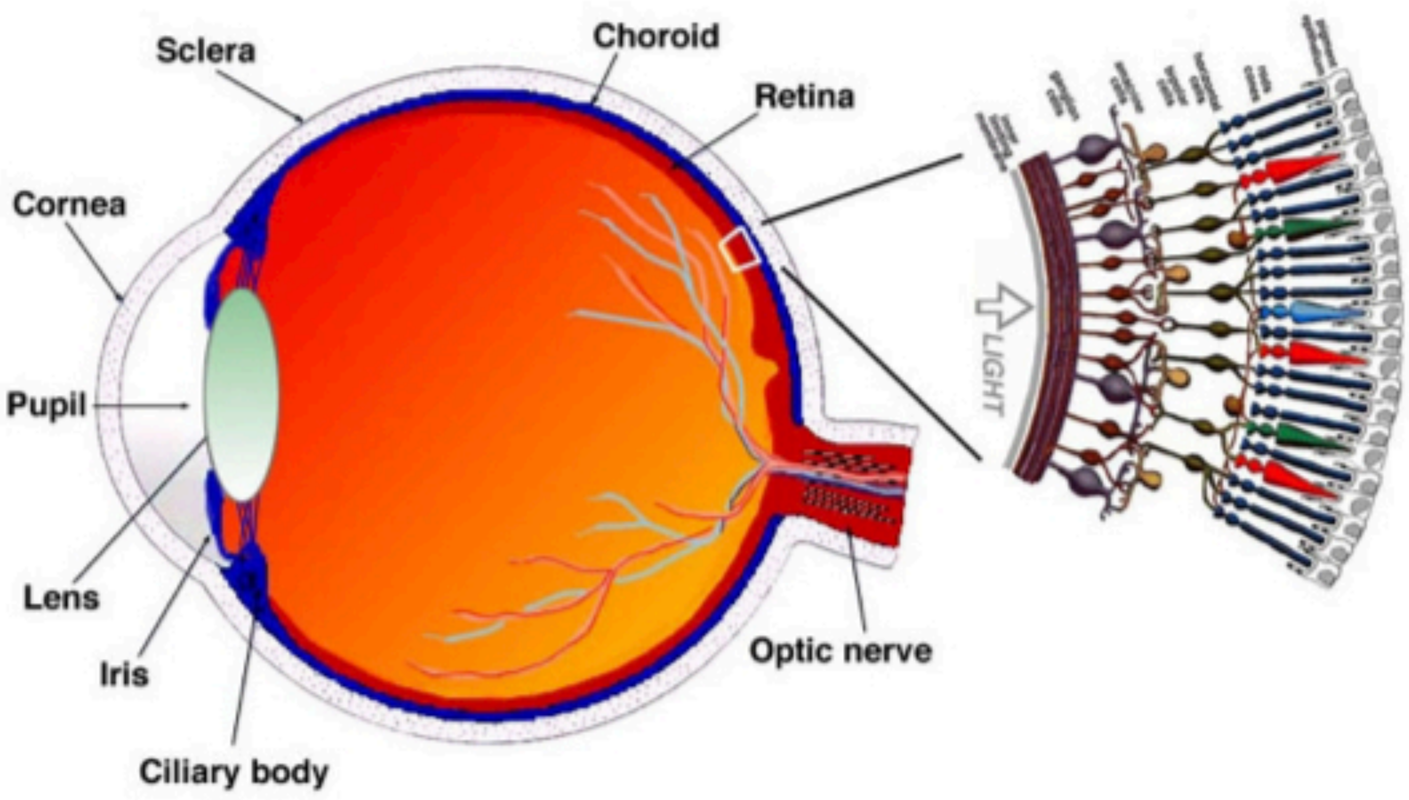
Sampling

Zoom in to see the binary pattern.



Sampling

Every image that we see is sampled by the retina, which contains $\approx$ 100 million rods and 6 million cones (average spacing $\approx 3\mu$m) which act as discrete sensors.



Courtesy of Helga Kolb, Eduardo Fernandez, and Ralph Nelson. Used with permission.

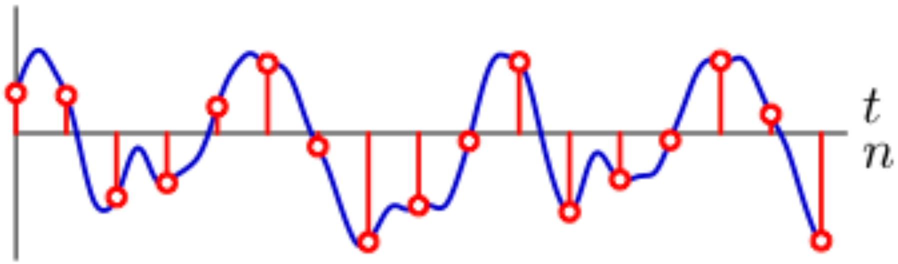
http://webvision.med.utah.edu/imageswv/sagschem.jpeg

Sampling and Reconstruction

To determine the effect of sampling, compare the original signal $x(t)$ to the signal $x_p(t)$ that is **reconstructed** from the samples $x[n]$.

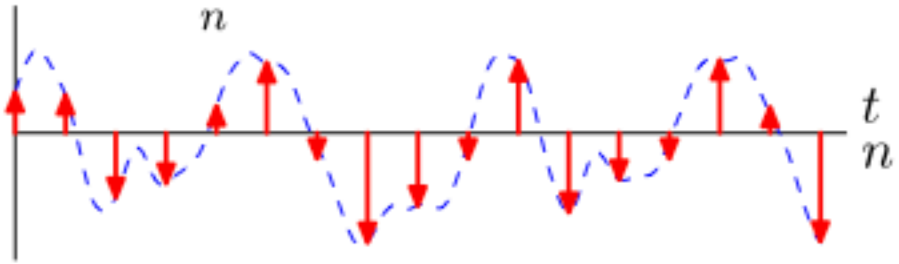
Uniform sampling (sampling interval T).

$$x[n] = x(nT)$$



Impulse reconstruction.

$$x_p(t) = \sum_n x[n]\delta(t - nT)$$



Reconstruction

Impulse reconstruction maps samples $x[n]$ (DT) to $x_p(t)$ (CT).

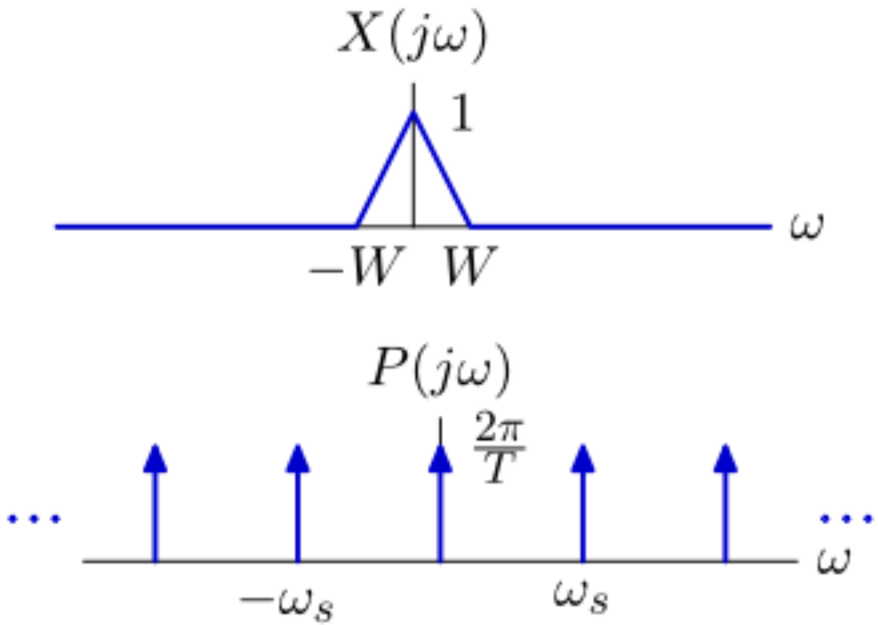
$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) \\ &= x(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\equiv\ p(t)} \end{aligned}$$

Resulting reconstruction $x_p(t)$ is equivalent to multiplying $x(t)$ by impulse train.

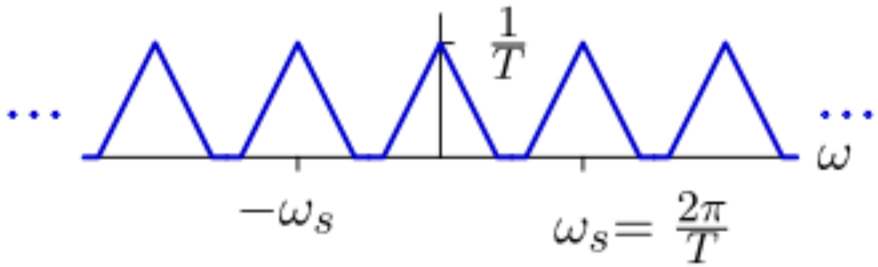
Sampling

Multiplication by an impulse train in time is equivalent to convolution by an impulse train in frequency.

→ generates multiple copies of original frequency content.



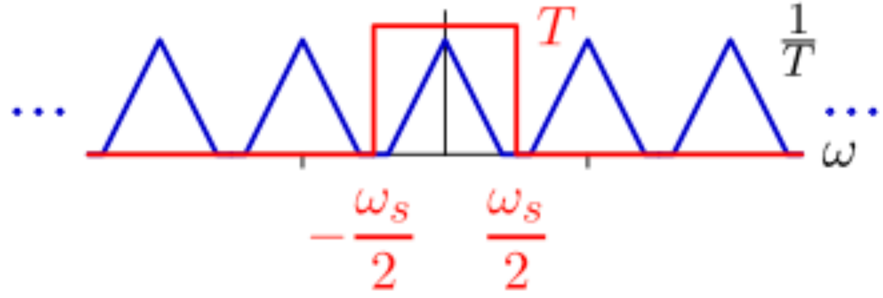
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Sampling

The high frequency copies can be removed with a low-pass filter (also multiply by T to undo the amplitude scaling).

$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Impulse reconstruction followed by ideal low-pass filtering is called **bandlimited reconstruction.**

The Sampling Theorem

If signal is bandlimited $\rightarrow$ sample without loosing information.

If $x(t)$ is bandlimited so that

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_m$$

then $x(t)$ is uniquely determined by its samples $x(nT)$ if

$$\omega_s = \frac{2\pi}{T} > 2\omega_m.$$

The minimum sampling frequency, $2\omega_m$, is called the "Nyquist rate."

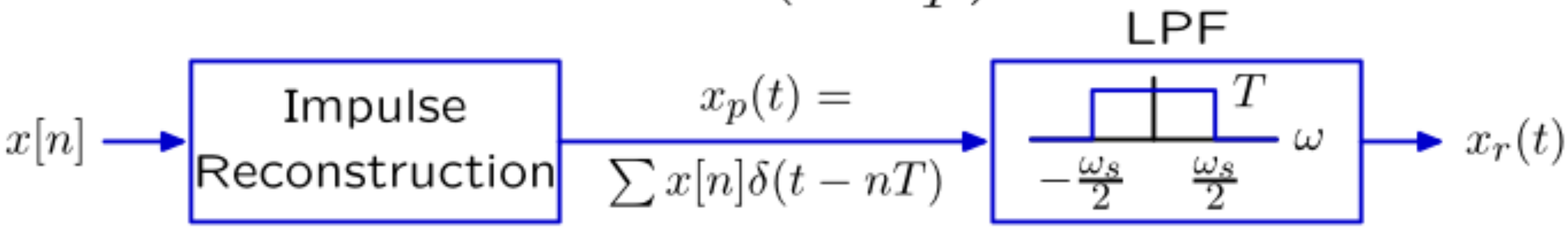
Summary

Three important ideas.

Sampling

$$x(t) \rightarrow x[n] = x(nT)$$

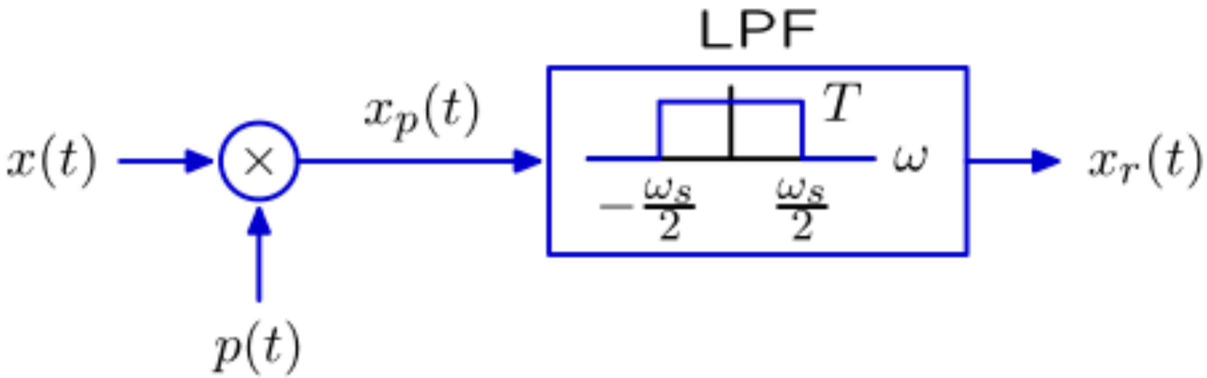
Bandlimited Reconstruction $\left(\omega_s = \frac{2\pi}{T}\right)$



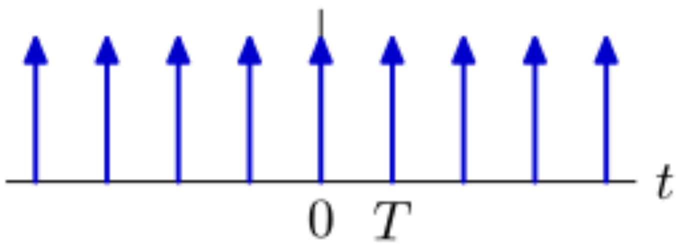
Sampling Theorem: If $X(j\omega) = 0 \;\forall\; |\omega| > \frac{\omega_s}{2}$ then $x_r(t) = x(t)$.

CT Model of Sampling and Reconstruction

Sampling followed by bandlimited reconstruction is equivalent to multiplying by an impulse train and then low-pass filtering.

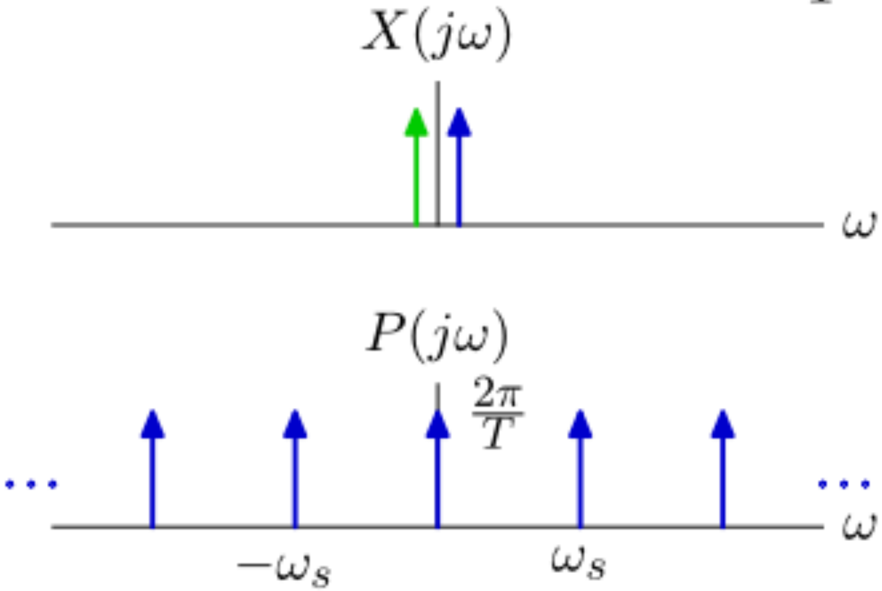


$p(t) =$ "sampling function"

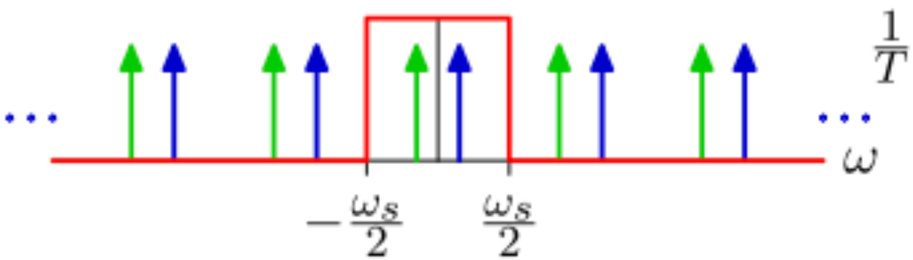


Aliasing

What happens if X contains frequencies $|\omega| > \frac{\pi}{T}$?



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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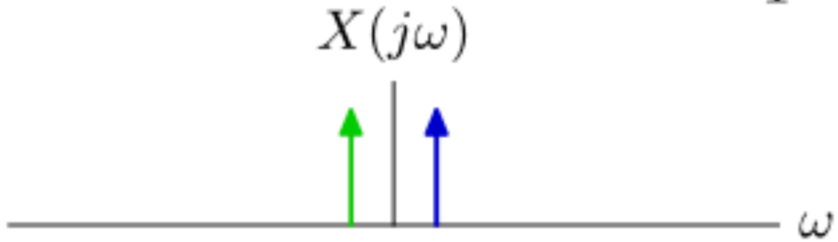
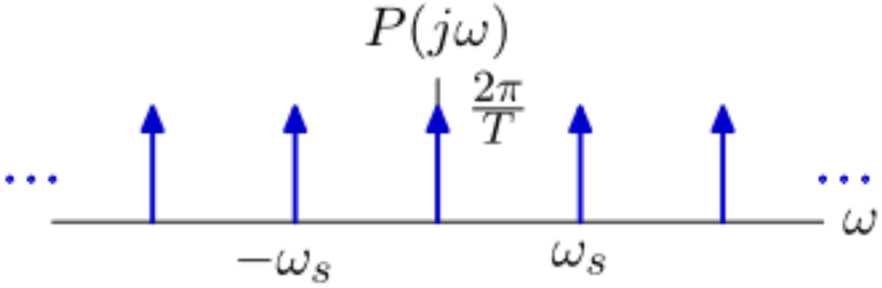
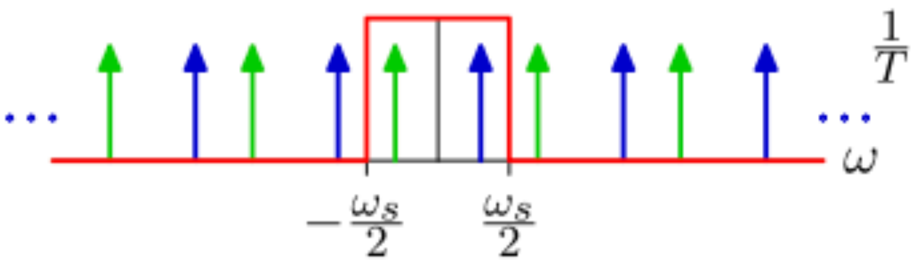




$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing

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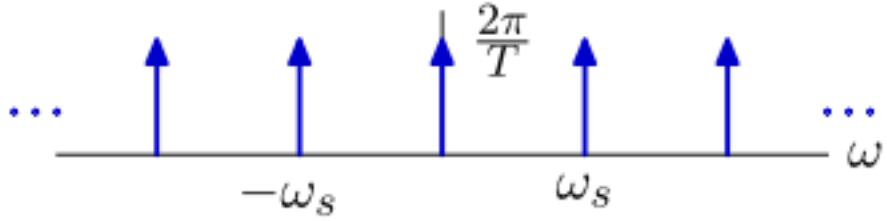

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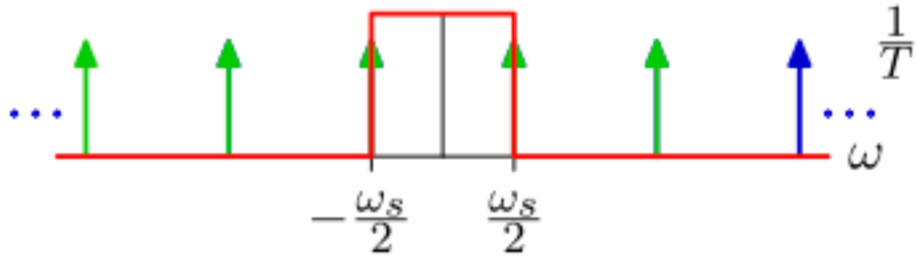

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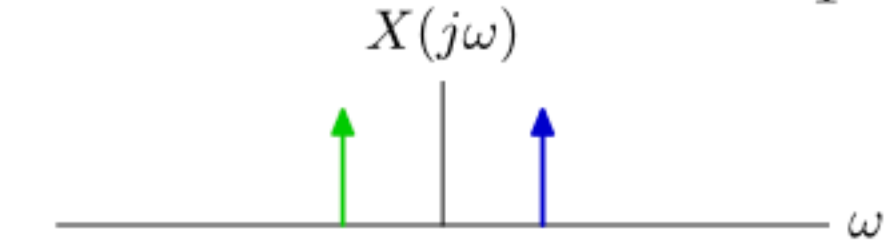
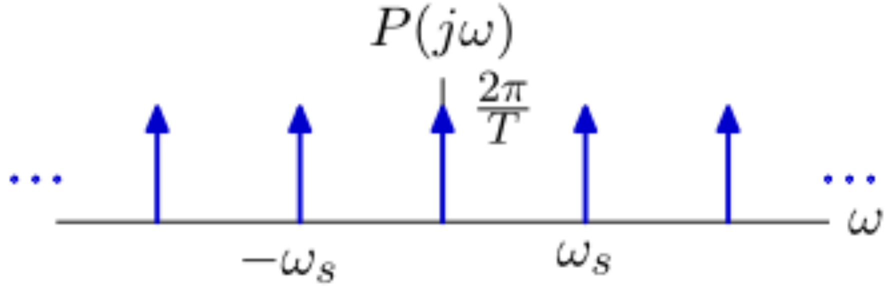
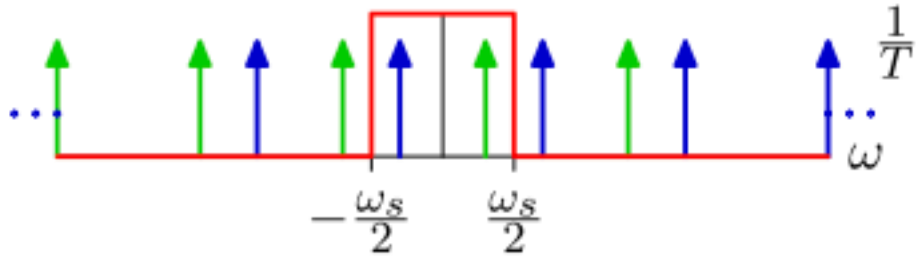


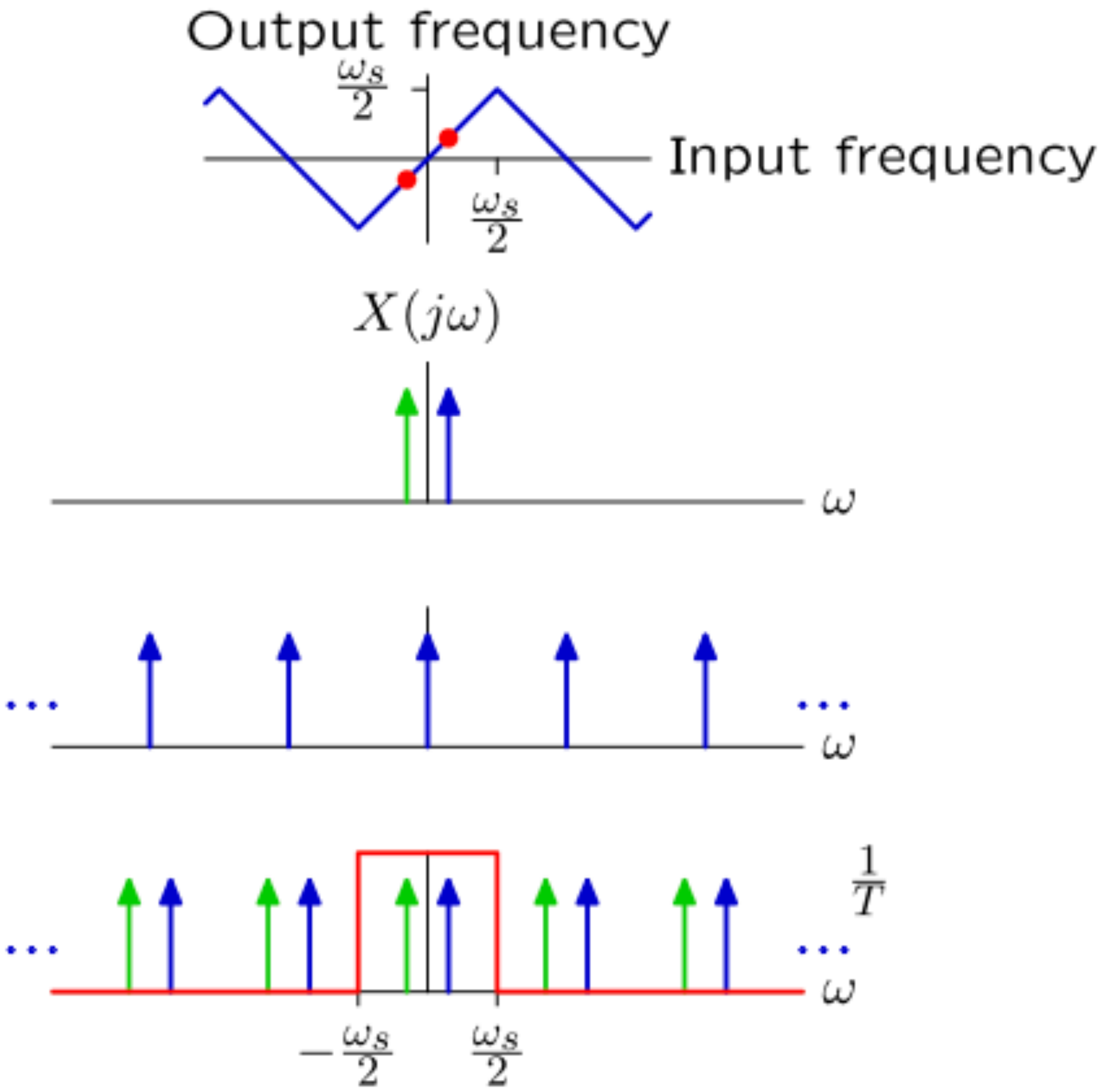


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing

The effect of aliasing is to wrap frequencies.



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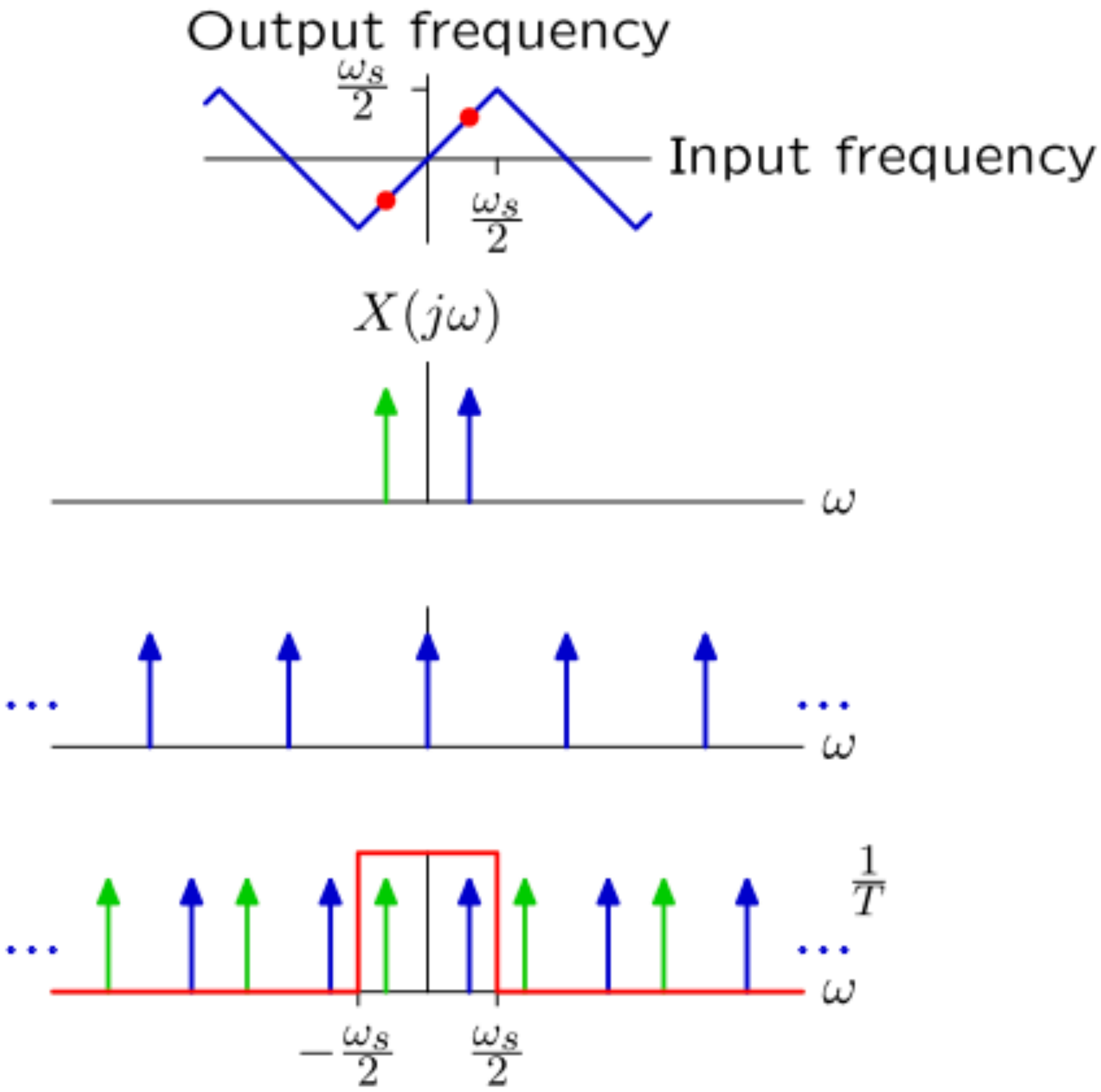

Aliasing

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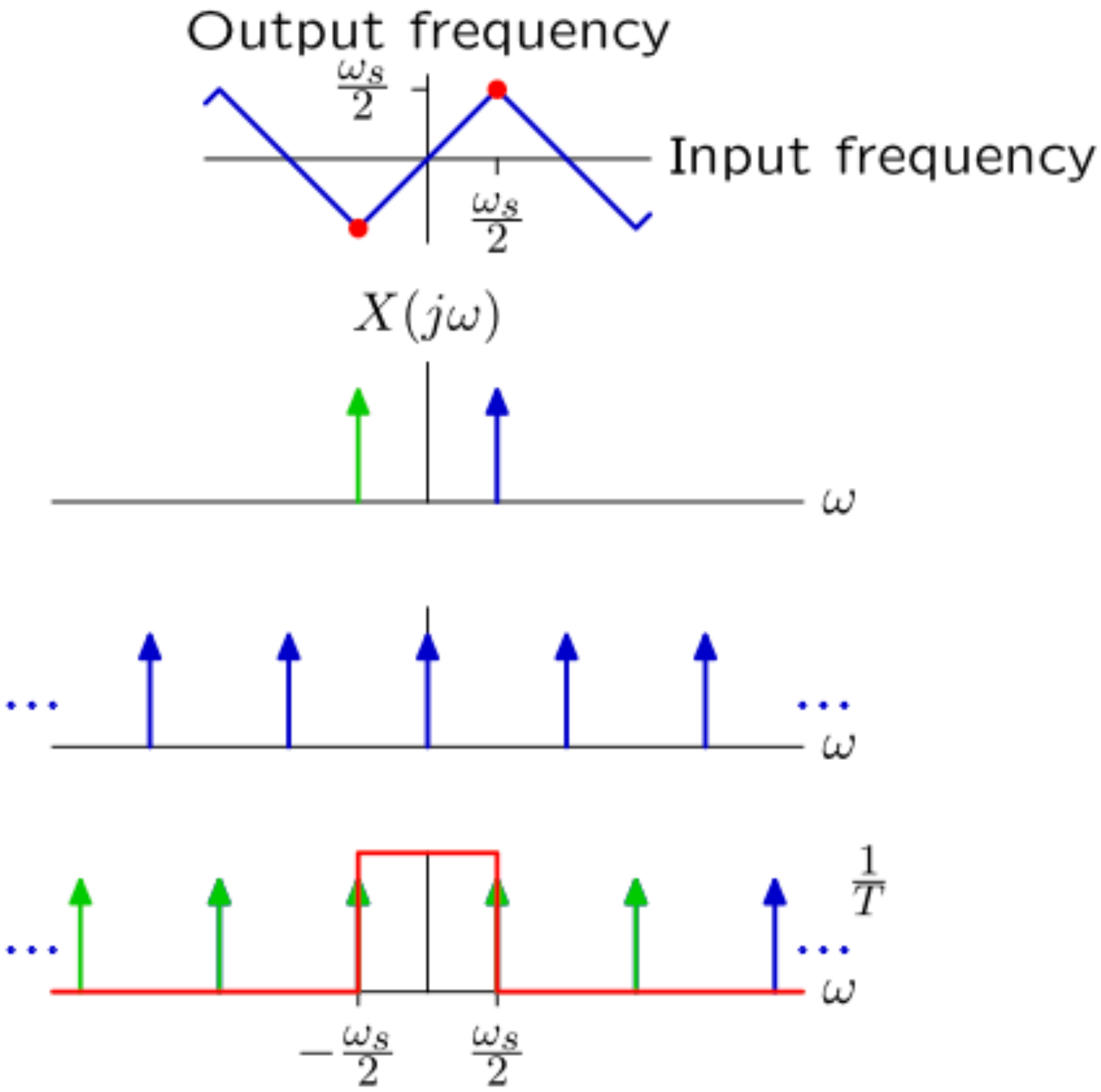

Aliasing

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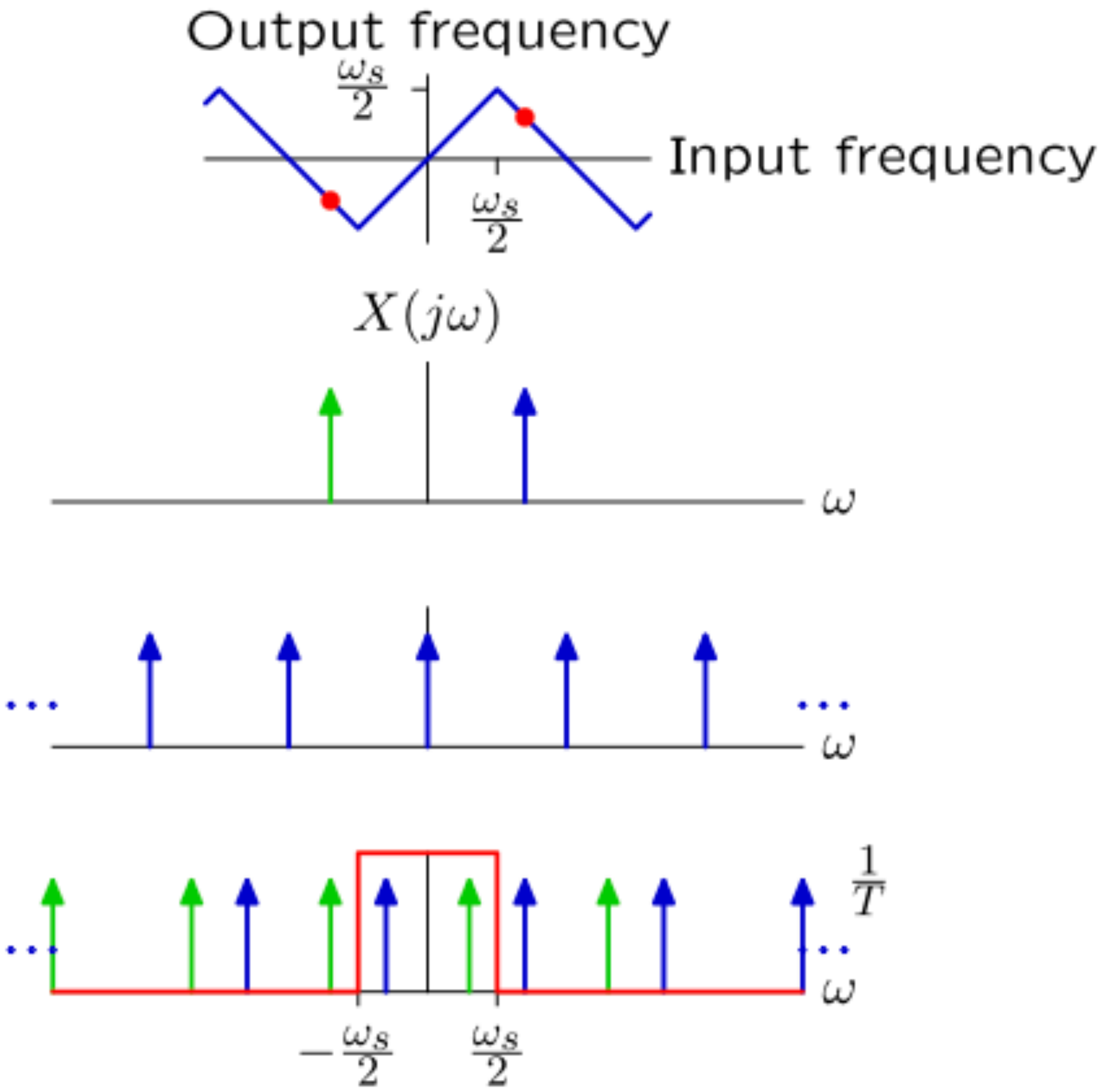
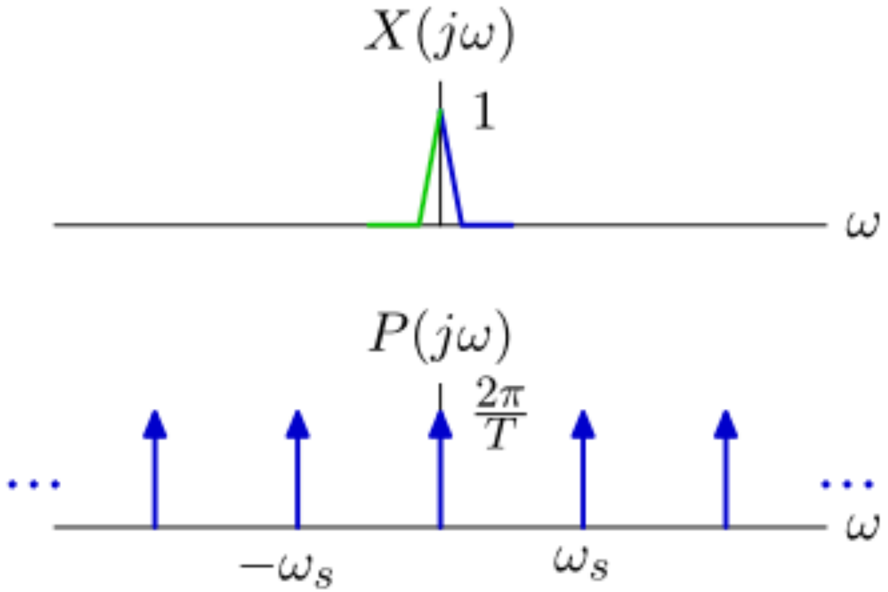
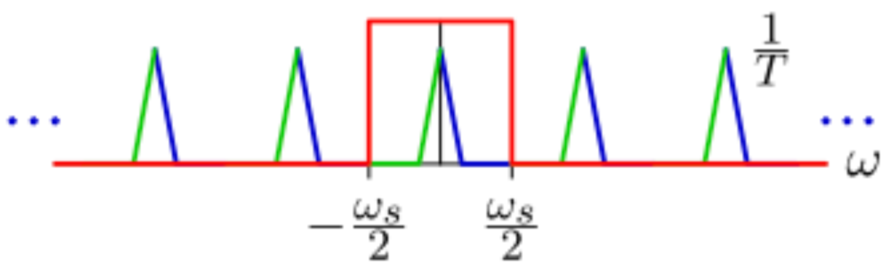


Aliasing

High frequency components of complex signals also wrap.



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing

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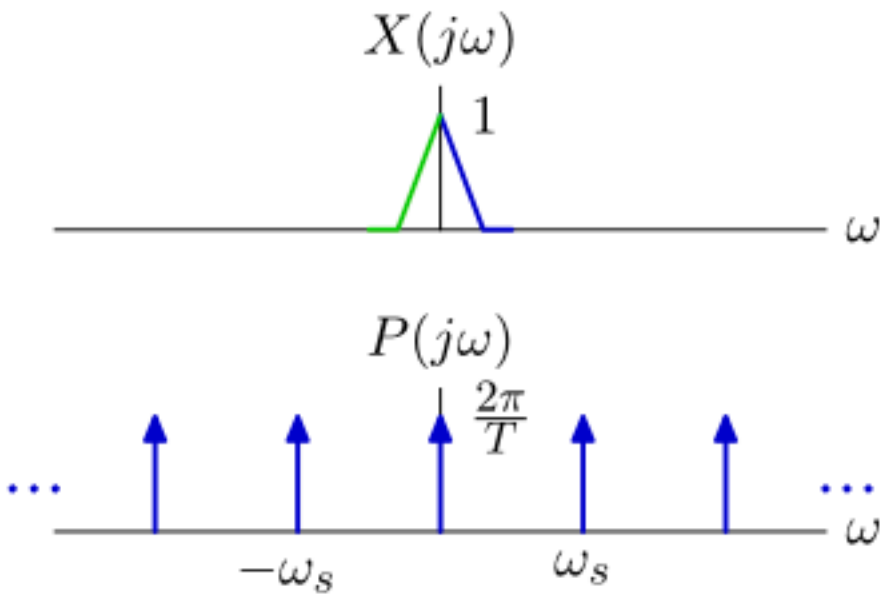
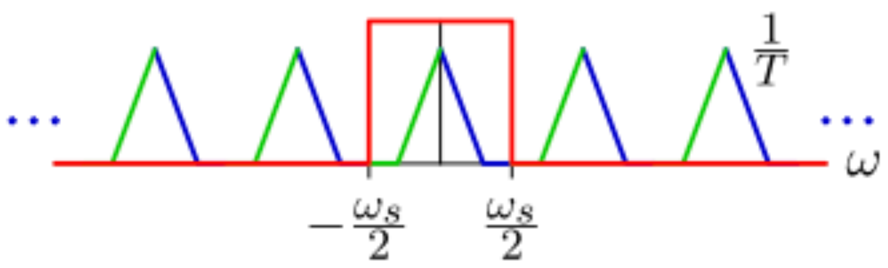


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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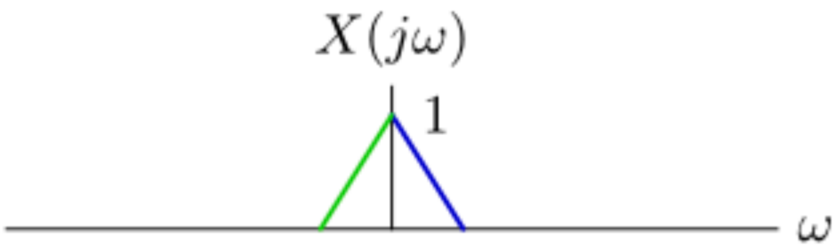
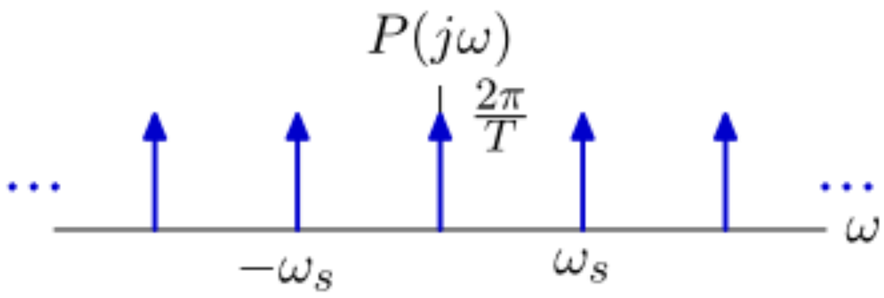
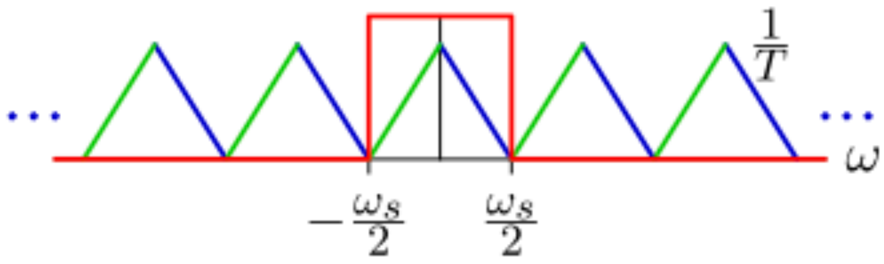




$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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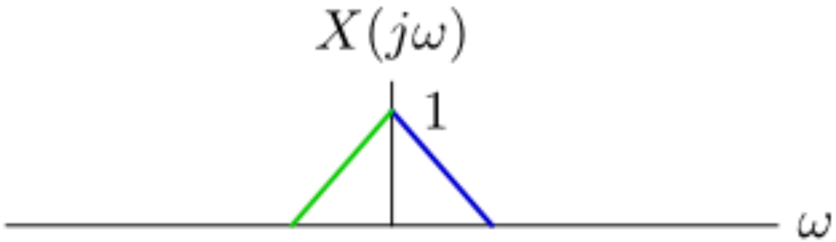

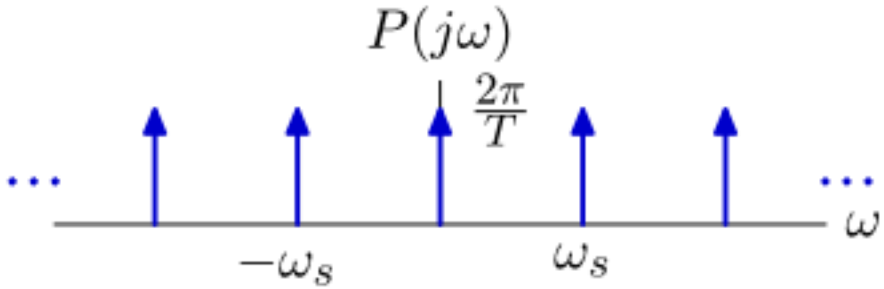

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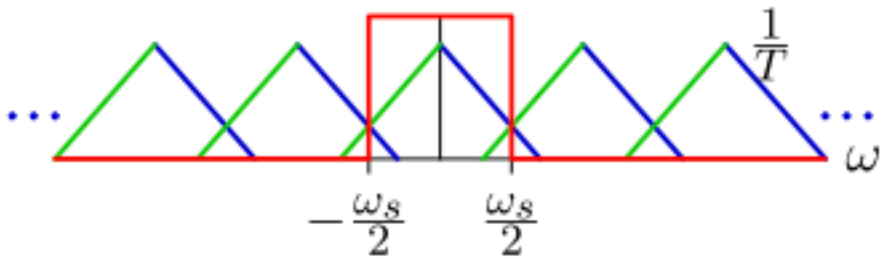
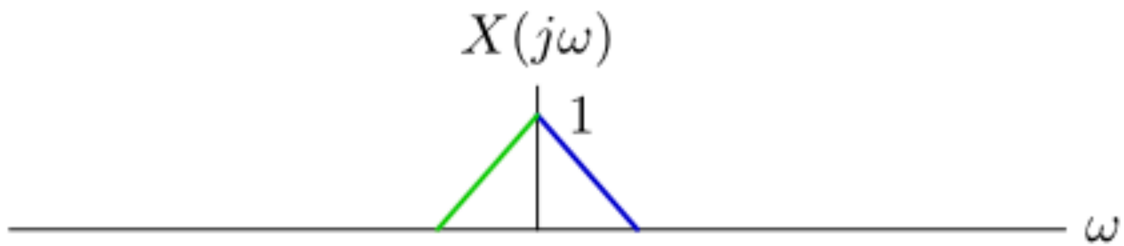
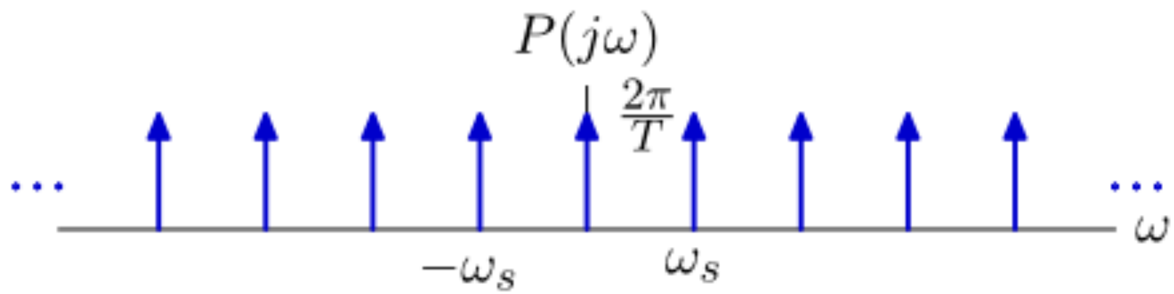


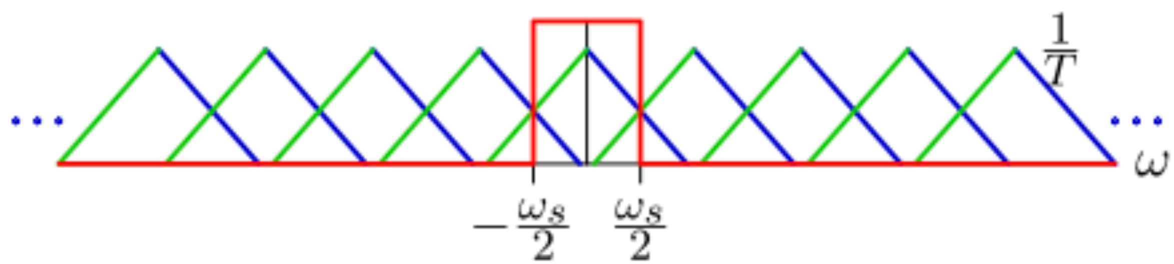
Aliasing

Aliasing increases as the sampling rate decreases.



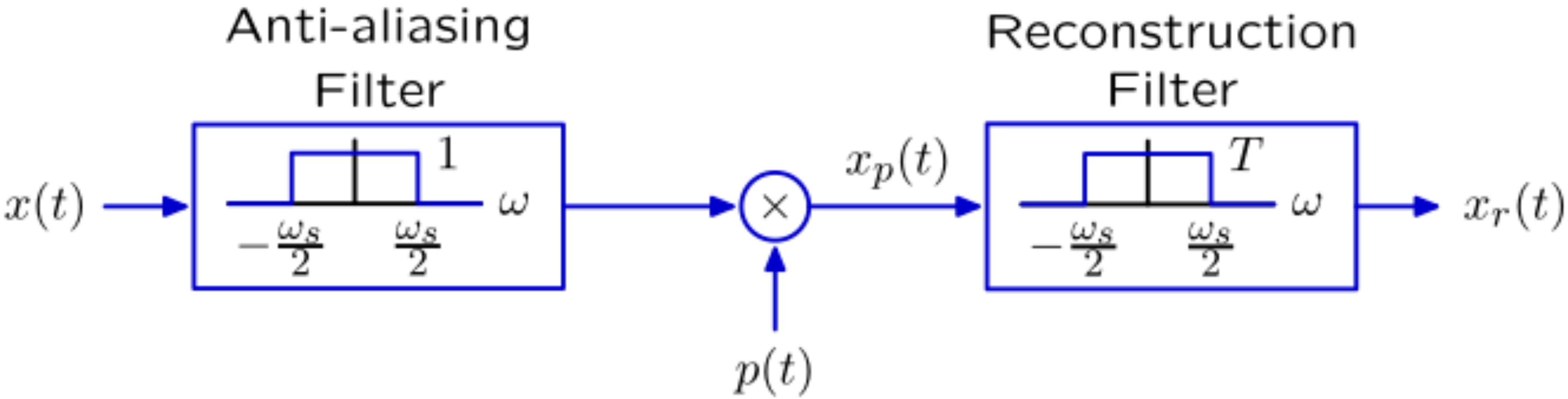


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Anti-Aliasing Filter

To avoid aliasing, remove frequency components that alias before sampling.



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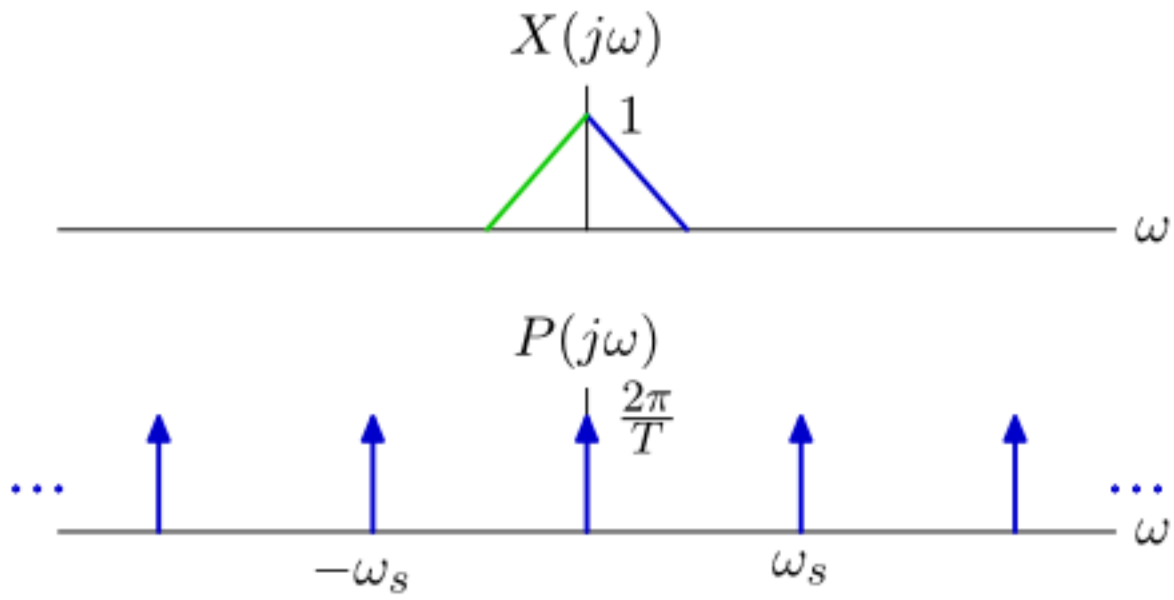
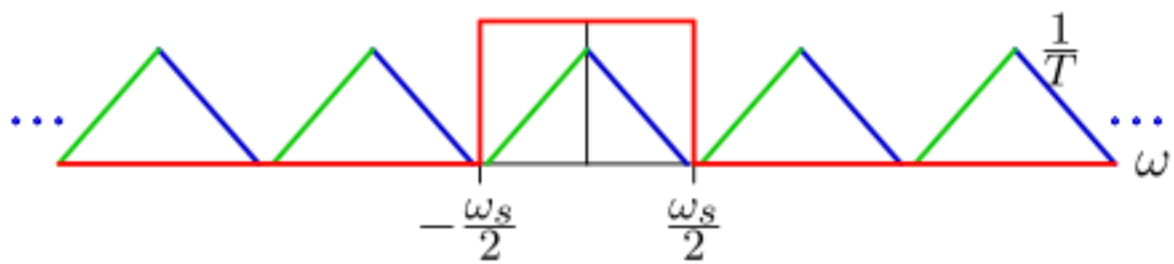
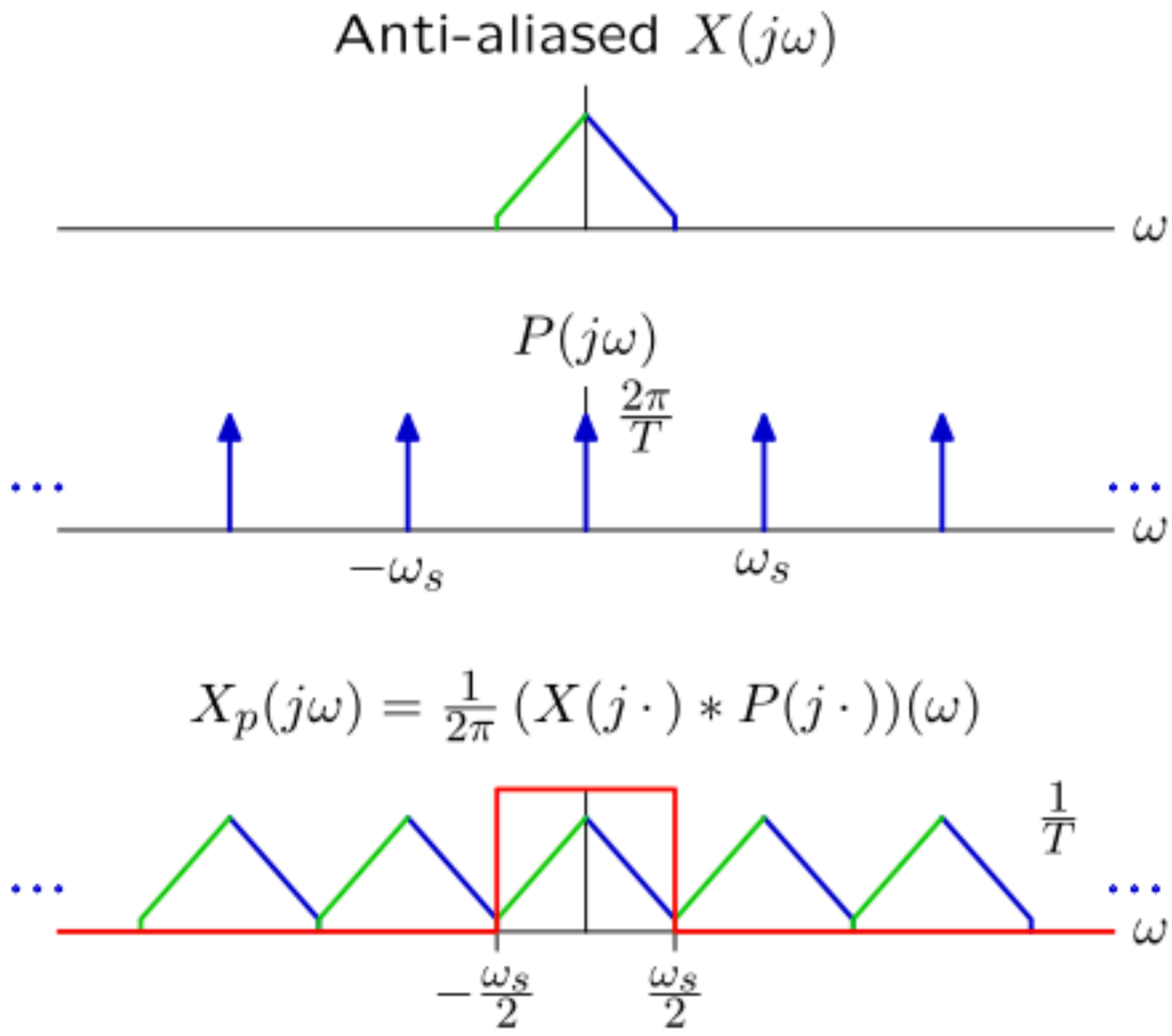


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing

Aliasing increases as the sampling rate decreases.



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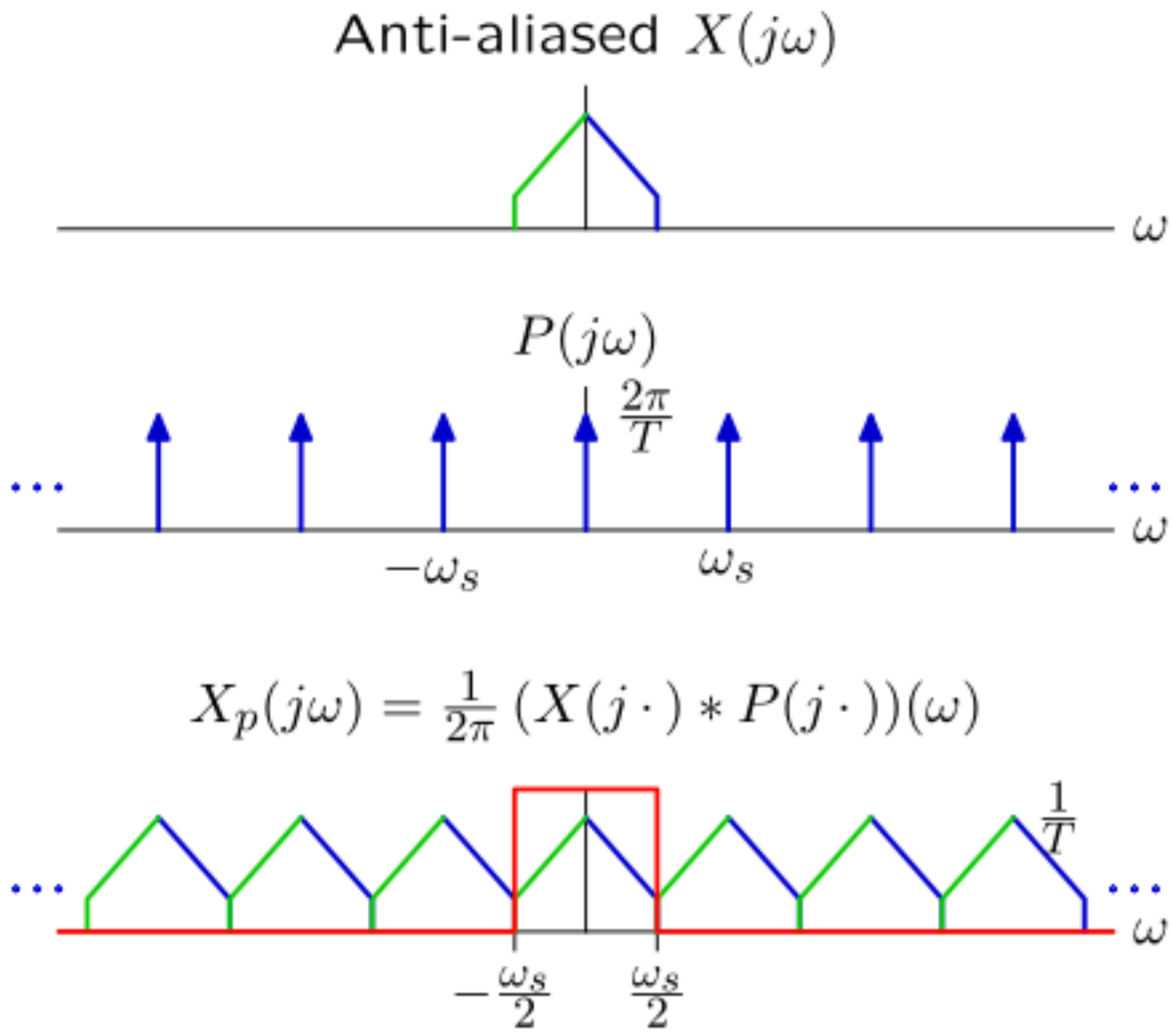

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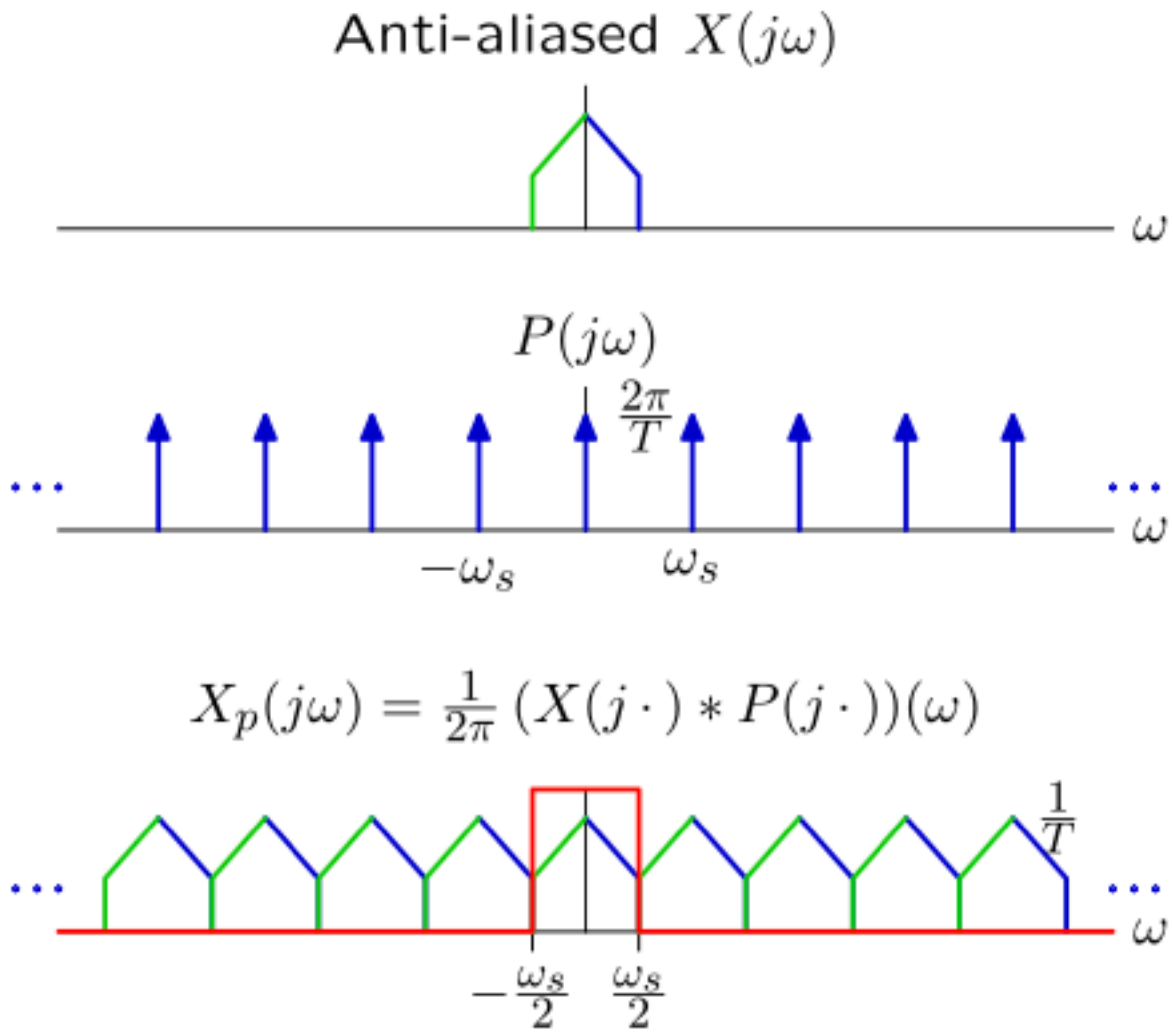

Aliasing

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3 × 3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Learning Goals

- What are the properties of an LTI system?
- What is the Sampling Theorem?
- What is Aliasing and how can we avoid it?